Continuous-variable probing of structured environments

Matteo Bina, Federico Grasselli and Matteo Paris

Università degli Studi di Milano

Aim of the work

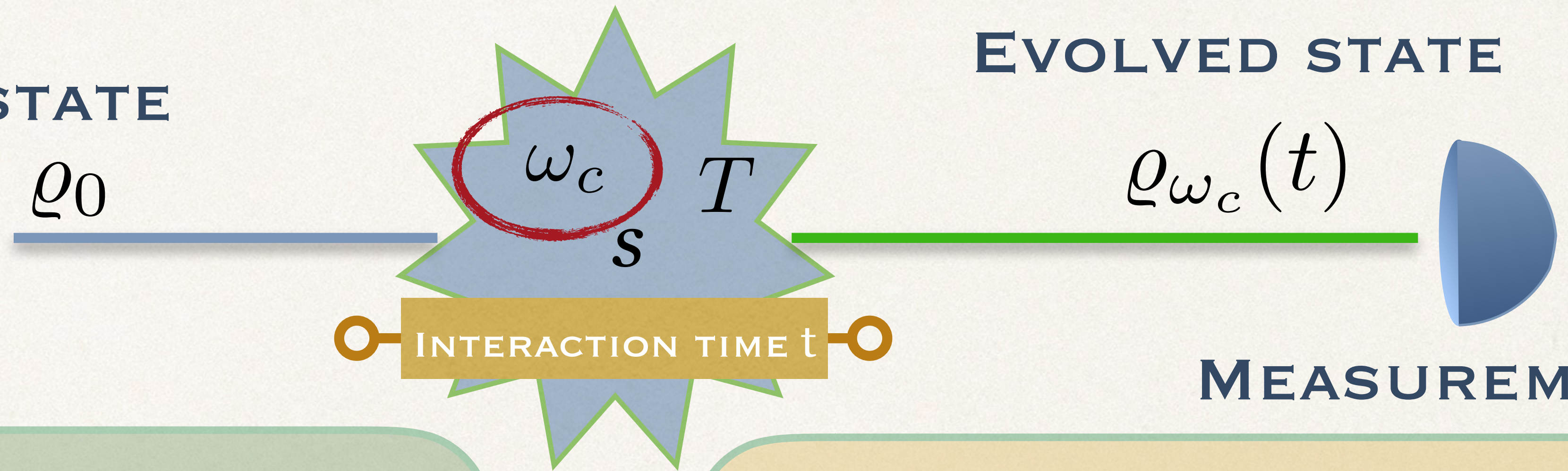
$\varrho_0 = S(\xi)\nu(n_{\rm th})S^\dagger(\xi)$

$\xi \quad n_{\rm th}$

STRUCTURED ENVIRONMENT

PROBE STATE

EVOLVED STATE



MEASUREMENT

Spectral density

$$J(\omega) = \omega_c \left(\frac{\omega}{\omega_c}\right)^s e^{-\frac{\omega}{\omega_c}}$$

- Ohmic $s = 1$
- sub-Ohmic $s = \frac{1}{2}$
- super-Ohmic $s = 3$

- Optimal POVM $\rightarrow$ QFI
- Feasible Measurement $\mathcal{O}$ $\rightarrow$ FI

Quantum Cramer-Rao bound

$$\mathrm{Var}_{\omega_c} \geq \frac{1}{F_{\mathcal{O}}(\omega_c)} \geq \frac{1}{H(\omega_c)}$$

The Model

✦ Hu, Paz, Zhang, PRD **45**, 2843 (1992)
✦ Vasile *et al.*, PRA **80**, 062324 (2009)

SYSTEM-ENVIRONMENT HAMILTONIAN

$$H = \frac{\hbar\omega_0}{2}\left(P^2 + X^2\right) + \sum_n \frac{\hbar\omega_n}{2}\left(P_n^2 + X_n^2\right) - \alpha X \otimes \sum_n \hbar\gamma_n X_n$$

SYSTEM MASTER EQUATION

Born (weak-coupling) approximation only! $\alpha \ll 1$

$$\frac{d}{dt}\varrho(t) = -\frac{\mathrm{i}}{\hbar}\left[H_0, \varrho(t)\right] + \mathrm{i}\, r(t)\left[X^2, \varrho(t)\right] - \mathrm{i}\,\gamma(t)\left[X, \{P, \varrho(t)\}\right] - \Delta(t)\left[X, [X, \varrho(t)]\right] + \Pi(t)\left[X, [P, \varrho(t)]\right]$$

$$r(t) = \alpha^2 \int_0^t d\tau \cos(\omega_0\tau) \int_0^\infty d\omega\, J(\omega) \sin(\omega\tau)$$

$$\gamma(t) = \alpha^2 \int_0^t d\tau \sin(\omega_0\tau) \int_0^\infty d\omega\, J(\omega) \sin(\omega\tau)$$

$$\Delta(t) = \alpha^2 \int_0^t d\tau \cos(\omega_0\tau) \int_0^\infty d\omega \coth\left(\frac{\hbar\omega}{2k_B T}\right) J(\omega) \cos(\omega\tau)$$

$$\Pi(t) = \alpha^2 \int_0^t d\tau \sin(\omega_0\tau) \int_0^\infty d\omega \coth\left(\frac{\hbar\omega}{2k_B T}\right) J(\omega) \cos(\omega\tau)$$

RESERVOIR SPECTRAL DENSITY

$$J(\omega) = \omega_c \left(\frac{\omega}{\omega_c}\right)^s \mathrm{e}^{-\frac{\omega}{\omega_c}}$$

The Model

Probe: single-mode Gaussian state

$$\varrho_0 = S(\xi)\nu(n_{\rm th})S^\dagger(\xi) \quad \Rightarrow \quad \chi[\vec{z}] = \exp\left[-\frac{1}{2}\vec{z}^T\Omega\,\sigma\,\Omega^T\vec{z} - \mathrm{i}\,\vec{z}^T\,\Omega\,\vec{\delta}\right]$$

characteristic function

Covariance Matrix

Null first-moment vector $\vec{\delta} = 0$

Secular approximation: neglect fast rotating terms $2\omega_0 t$

$$\sigma_0 = \left(\frac{1}{2} + n_{\rm th}\right)\begin{pmatrix} e^{2\xi} & 0 \\ 0 & e^{-2\xi} \end{pmatrix} \quad \Rightarrow \quad \sigma(t) = \Delta_\Gamma(t)\,\mathbb{I} + e^{-\Gamma(t)}R(t)\sigma_0 R^{-1}(t)$$

Evolved probe state depends on:

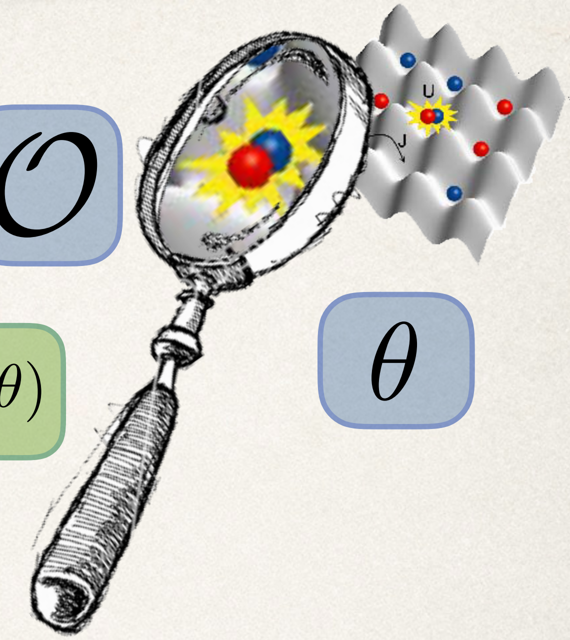
- squeezing ξ
- initial thermal photons $n_{\rm th}$
- interaction time t
- reservoir temperature T

$$\Delta_\Gamma(t) \equiv e^{-\Gamma(t)}\int_0^t e^{\Gamma(\tau)}\Delta(\tau)d\tau$$

$$\Gamma(t) = 2\int_0^t \gamma(\tau)d\tau$$

$$R(t) \approx \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix}$$

Q.E.T. Quantum Estimation Theory



$\mathcal{O}$ θ

Estimate a non-directly observable parameter θ :

Observable $\mathcal{O}$ → Set of data $x = \{x_1, \ldots, x_M\}$ → Sample probability $p(x|\theta)$

Mean value $\mathrm{E}_\theta[\hat{\theta}] = \int d\,x_1 \ldots d\,x_M\, p(x_1, \ldots, x_M|\theta)\, \hat{\theta}(x_1, \ldots, x_M)$

Variance $\mathrm{Var}[\hat{\theta}] = E\left[\left(\hat{\theta} - \mathrm{E}_\theta[\hat{\theta}]\right)^2\right] = \int d\,x_1 \ldots d\,x_M\, p(x_1, \ldots, x_M|\theta)\, \left(\hat{\theta} - \mathrm{E}_\theta[\hat{\theta}]\right)^2$

Fisher Information (FI)

$$F_{\mathcal{O}}(\theta) = \int d\,x\, \frac{\left[\partial_\theta p(x|\theta)\right]^2}{p(x|\theta)}$$

Cramer-Rao bound

$$\mathrm{Var}_\theta \geq \frac{1}{F_{\mathcal{O}}(\theta)}$$

Quantum Fisher Information (QFI)

$H(\theta)$ Optimized over all measurements

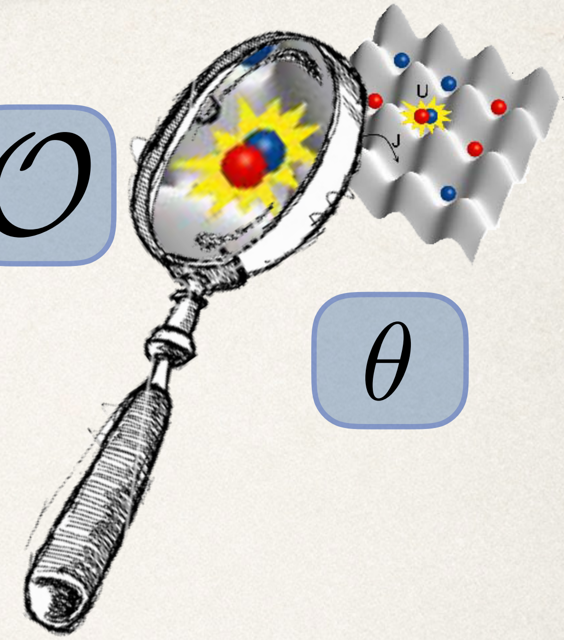
Quantum Cramer-Rao bound

$$\mathrm{Var}_\theta \geq \frac{1}{F_{\mathcal{O}}(\theta)} \geq \frac{1}{H(\theta)}$$

Q.E.T. Quantum Estimation Theory

✦ Z. Jiang, Phys. Rev. A **89**, 032128 (2014)

$\mathcal{O}$



θ

Gaussian States with $\vec{\delta} = 0$

$$H(\theta) = \frac{1}{2\lambda^4 - 1/8}\left\{\lambda^4 \mathrm{Tr}[(\sigma^{-1}\dot{\sigma})^2] - \frac{1}{4}\mathrm{Tr}[(\Omega\dot{\sigma})^2]\right\}$$

The parameter of interest is the CUTOFF FREQUENCY $\theta \rightarrow \omega_c$

Fisher Information (FI)

$$F_{\mathcal{O}}(\theta) = \int dx\, \frac{[\partial_\theta p(x|\theta)]^2}{p(x|\theta)}$$

Cramer-Rao bound

$$\mathrm{Var}_\theta \geq \frac{1}{F_{\mathcal{O}}(\theta)}$$

Quantum Fisher Information (QFI)

$H(\theta)$ Optimized over all measurements

Quantum Cramer-Rao bound

$$\mathrm{Var}_\theta \geq \frac{1}{H(\theta)} \geq \frac{1}{F_{\mathcal{O}}(\theta)}$$

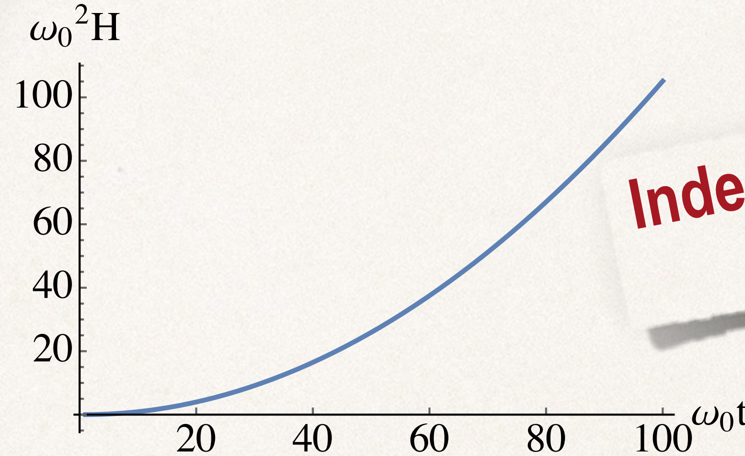
Results: QFI

Reservoir Temperature

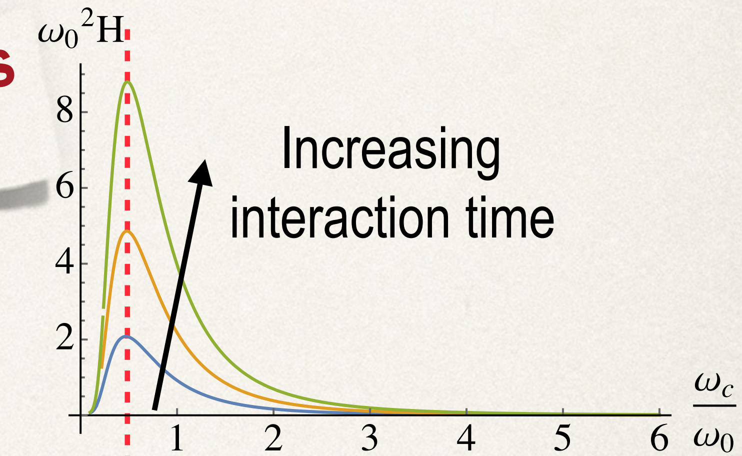
QFI : time

QFI : cutoff/probe frequency

HIGH - T



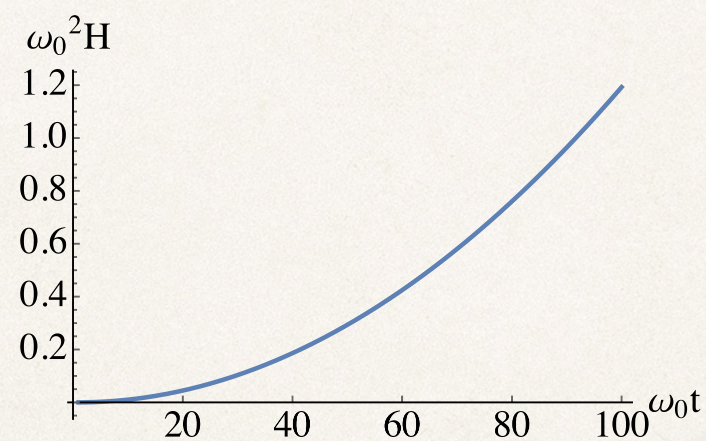
Independent on reservoir s and time t !!

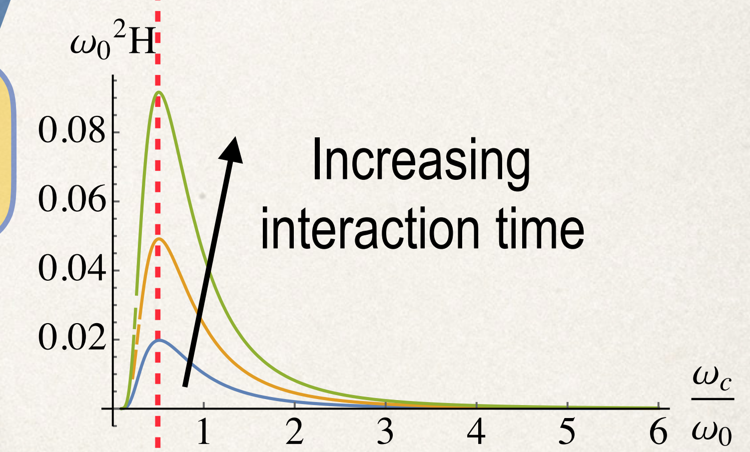


Scaling factor

$$\frac{\mathrm{QFI}^{\mathrm{H}}}{\mathrm{QFI}^{\mathrm{L}}} = f(\xi, n_{\mathrm{th}}, T)$$

LOW - T

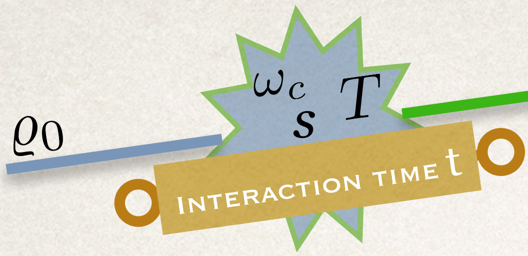




$$\left(\frac{\omega_c}{\omega_0}\right)^{(\max)} = f(s) = \begin{cases} (s \pm \sqrt{s})^{-1} & \text{for } s \neq 0\,,\ s \neq 1 \\ 1/2 & \text{for } s = 1\,. \end{cases}$$

Independent on temperature T and time t!!

$$\left(\frac{\omega_c}{\omega_0}\right)^{(\max)}$$



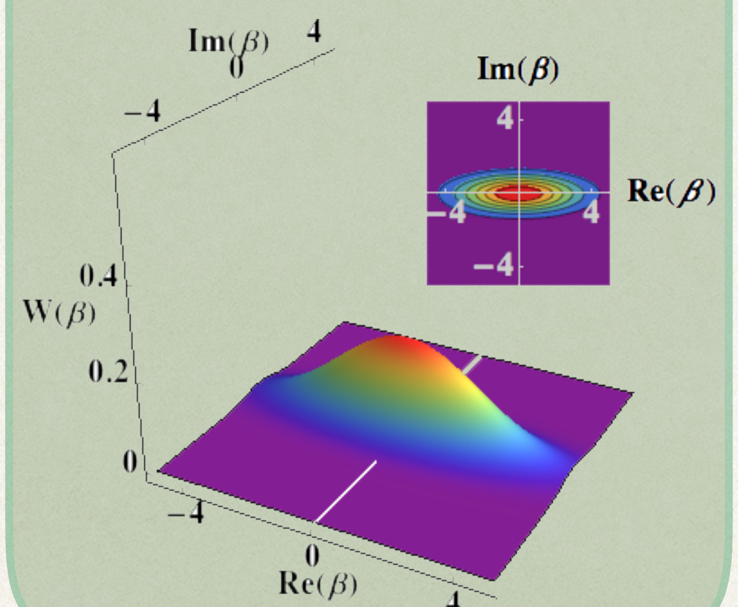
Results: QFI

QFI : initial state parameters $\xi \quad n_{\mathrm{th}}$

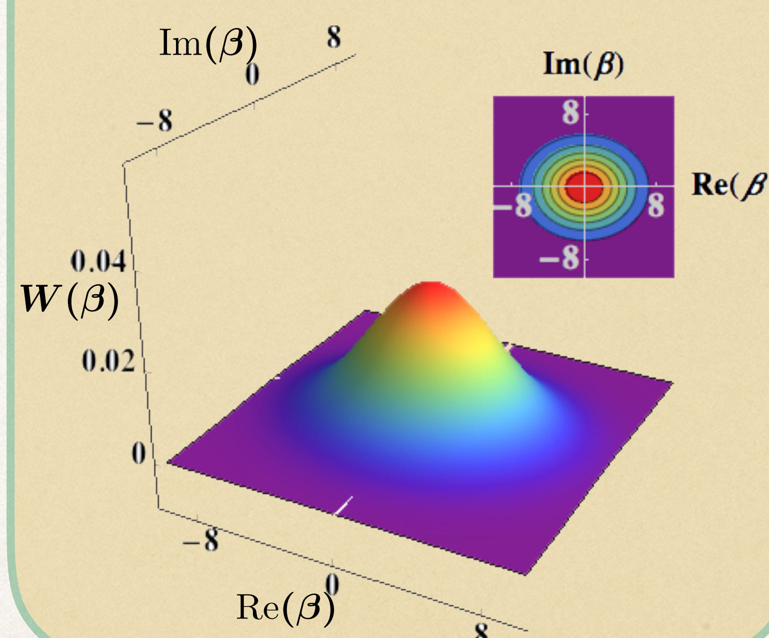
Squeezed thermal states :

$$\varrho_0 = S(\xi)\nu(n_{\mathrm{th}})S^\dagger(\xi)$$

$\xi = 0.4$
$n_{\mathrm{th}} = 4$



$\xi = 0.1$
$n_{\mathrm{th}} = 10$



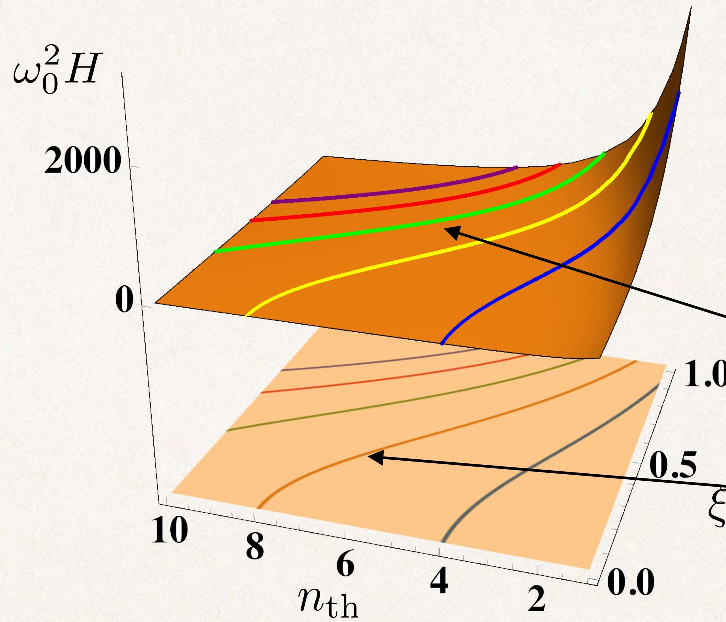
Results: QFI

$$\varrho_0 = S(\xi)\nu(n_{\rm th})S^\dagger(\xi)$$

QFI : INITIAL STATE PARAMETERS

$\xi \quad n_{\rm th}$

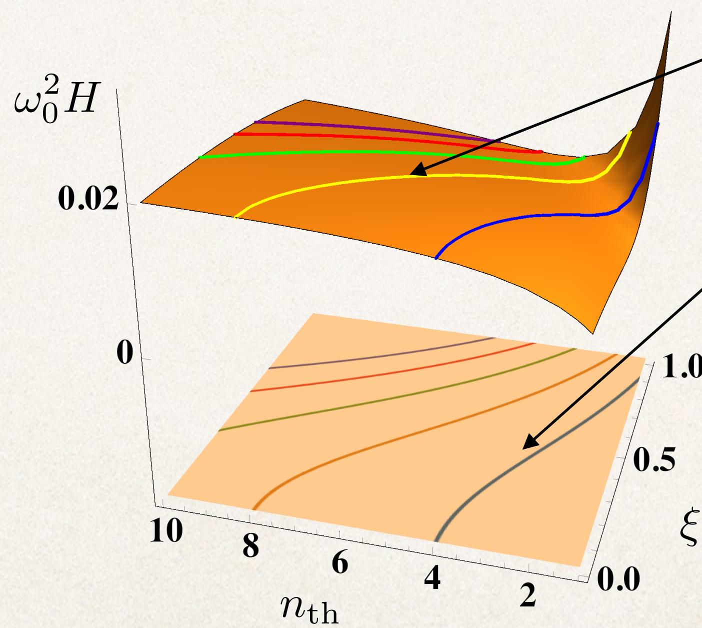
HIGH - T



Total energy

$$\mathcal{N} = \langle a^\dagger a\rangle_{\rho_0} = \sinh^2(\xi) + n_{\rm th}\,\left[1 + 2\sinh^2(\xi)\right]$$

LOW - T



Fixed probe total energy $\mathcal{N}$

QFI ↑ ξ ↑

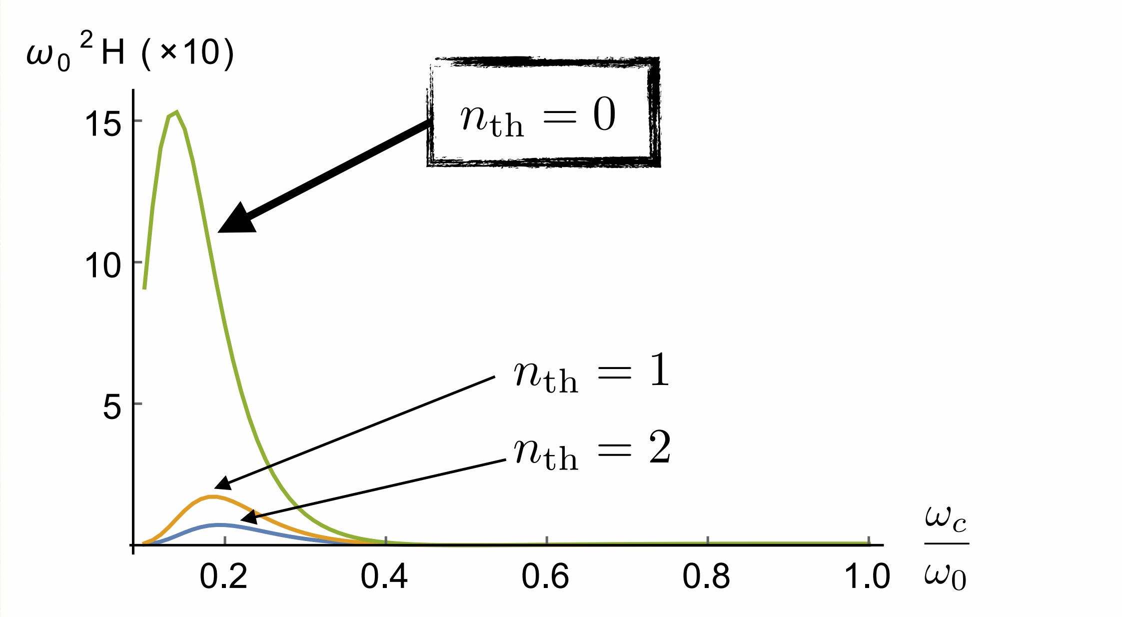
$n_{\rm th}$ ↓

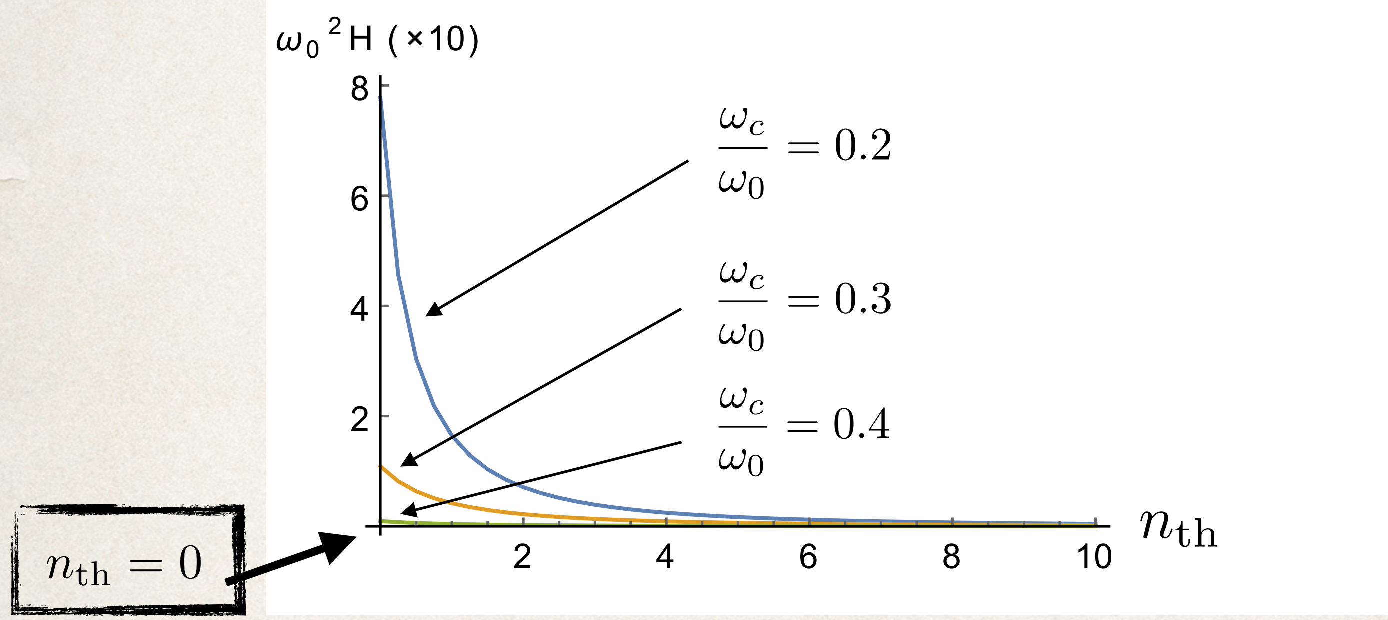
Results: QFI

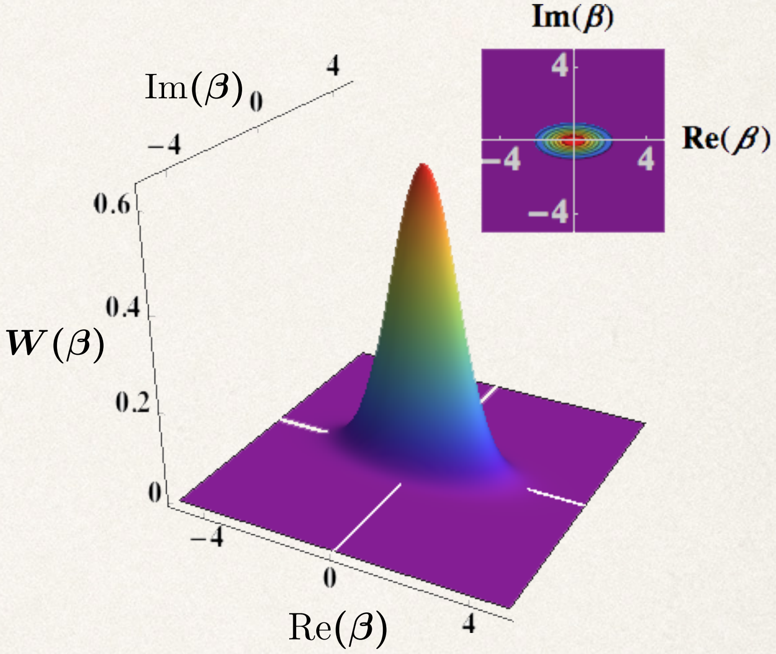
$$\rho_0 = S(\xi)|0\rangle\langle 0|S^\dagger(\xi)$$

QFI : INITIAL SQUEEZED VACUUM STATE

$n_{\rm th} = 0$



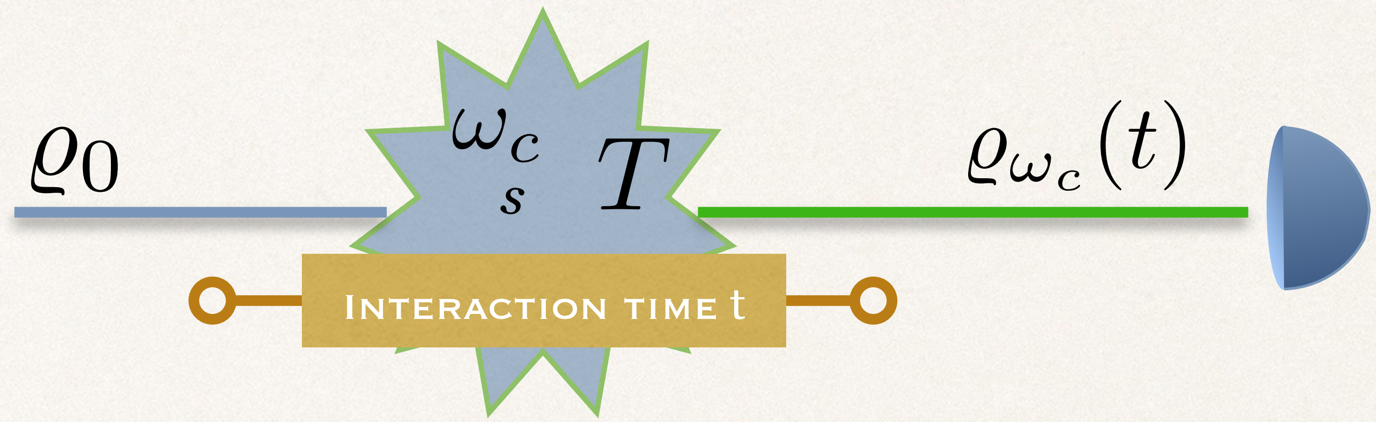




These plots are obtained for:

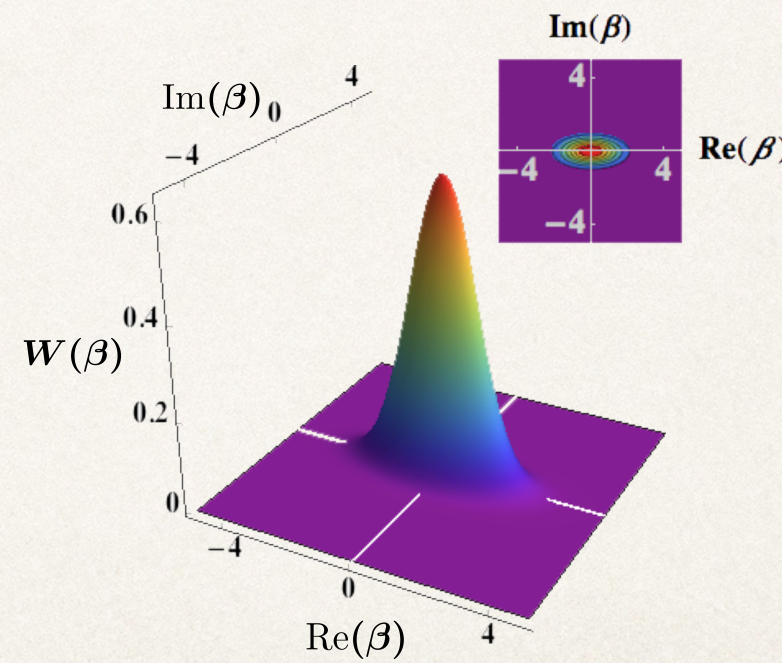
- Super-Ohmic reservoir (s = 3)
- Squeezing ξ = 0.1
- High Temperature limit

Results: QFI



QFI : Best strategy is to probe with a single-mode SQUEEZED VACUUM state

$$\rho_0 = S(\xi)|0\rangle\langle 0|S^{\dagger}(\xi)$$



Results: FI

Fisher Information (FI)

$$F_{\mathcal{O}}(\theta) = \int d\,x \, \frac{\left[\partial_\theta p(x|\theta)\right]^2}{p(x|\theta)}$$

$$\theta \rightarrow \omega_c$$

Quantum Cramer-Rao bound

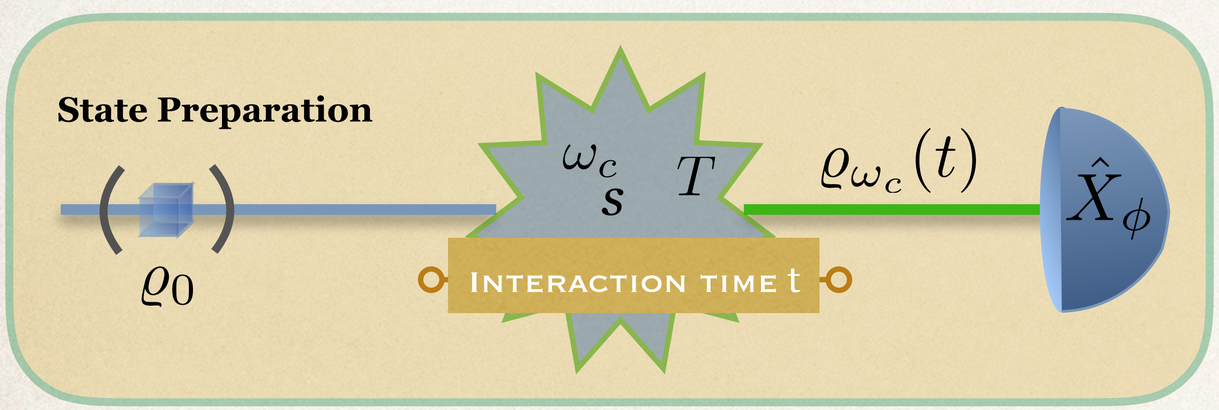
$$\mathrm{Var}_\theta \geq \frac{1}{F_{\mathcal{O}}(\theta)} \geq \frac{1}{H(\theta)}$$

HOMODYNE DETECTION

$$\mathcal{O} \rightarrow \hat{X}_\phi$$

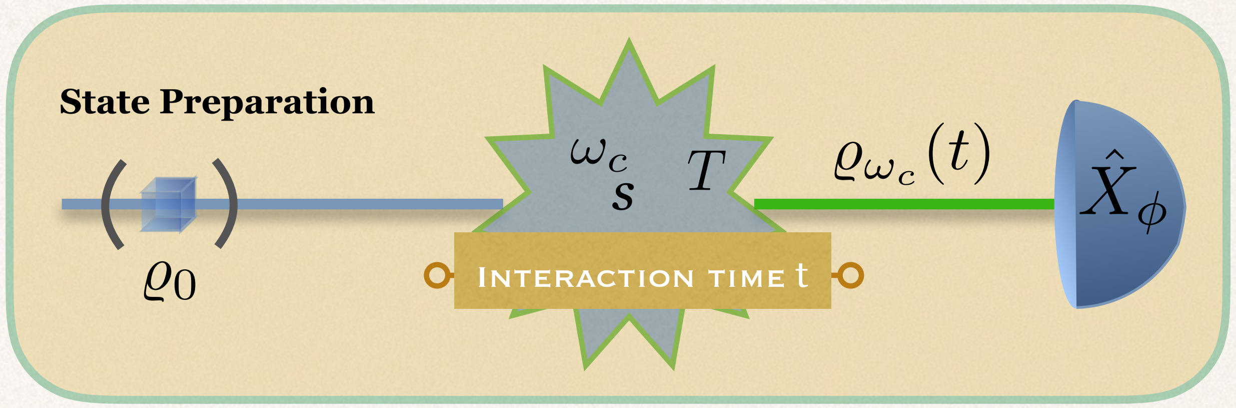
$\hat{X}_0$ **Position**

$\hat{X}_{\frac{\pi}{2}} = \hat{P}$ **Momentum**

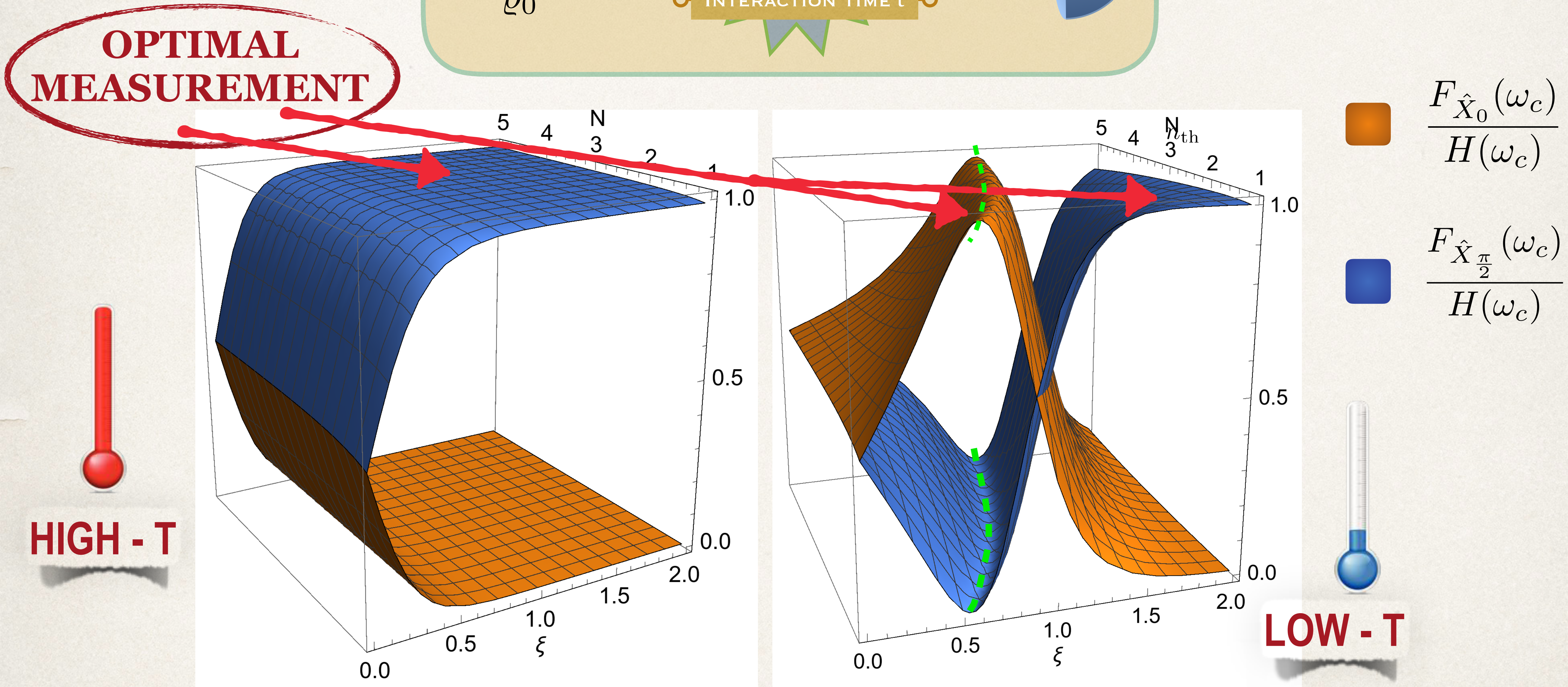


$$\hat{X}_\phi = \frac{\mathrm{e}^{-\mathrm{i}\phi} a + \mathrm{e}^{\mathrm{i}\phi} a^\dagger}{\sqrt{2}}$$

Results: FI



State Preparation
ϱ_0
ω_c
s
T
$\varrho_{\omega_c}(t)$
$\hat{X}_\phi$
Interaction time t



OPTIMAL MEASUREMENT
$\frac{F_{\hat{X}_0}(\omega_c)}{H(\omega_c)}$
$\frac{F_{\hat{X}_{\frac{\pi}{2}}}(\omega_c)}{H(\omega_c)}$
N
$\bar{n}_{th}$
ξ
HIGH - T
LOW - T

Conclusions

THNAKS FOR YUOR ATTENIOTN!

Highest QFI for the cutoff frequency?

$$\varrho_0 = S(\xi)\nu(n_{\mathrm{th}})S^\dagger(\xi)$$

SQUEEZED VACUUM

$$\rho_0 = S(\xi)|0\rangle\langle 0|S^\dagger(\xi)$$

Invariants and scaling factors of the QFI?

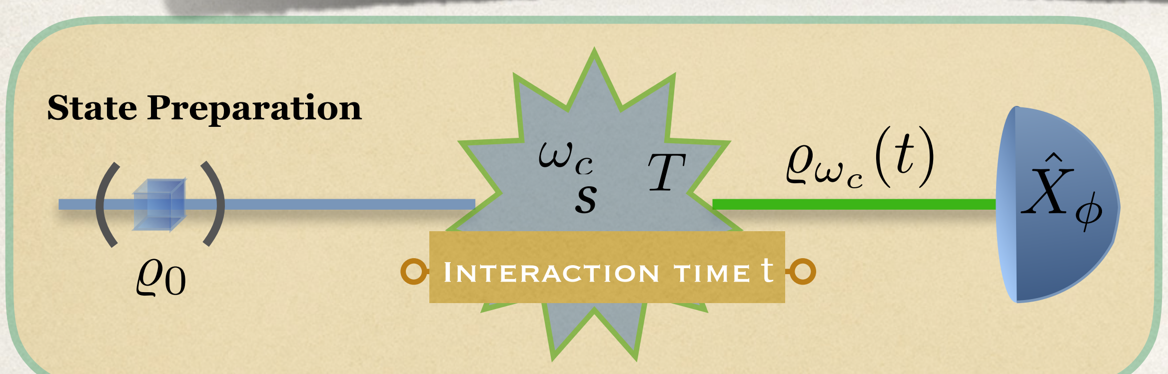
$$\left(\frac{\omega_c}{\omega_0}\right)^{(\max)}$$

Independent on temperature T and time t!!

$$\frac{\mathrm{QFI}^{\mathrm{H}}}{\mathrm{QFI}^{\mathrm{L}}} = f(\xi, n_{\mathrm{th}}, T)$$

Independent on reservoir s and time t !!

Optimal measurement schemes?



HOMODYNE DETECTION

$$\mathrm{Var}_{\omega_c} \geq \frac{1}{F_{\mathcal{O}}(\omega_c)} \geq \frac{1}{H(\omega_c)}$$