Memory kernels for collisional models

Bassano Vacchini

Università degli Studi di Milano
Dipartimento di Fisica
&
INFN

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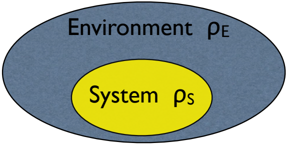
QuProCS II

Open quantum systems

- Quantum system interacting with environment

[Davies, 1976; Spohn, 1980; Alicki & Lendi, 1987; Weiss, 1999; Holevo, 2001; Breuer & Petruccione, 2002]



- Bipartite setting $H \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_E)$ $\rho_{SE} \in \mathcal{T}(\mathcal{H}_S \otimes \mathcal{H}_E)$
  System observables only determined by $\rho_S(t) = \mathrm{Tr}_E\, \rho_{SE}(t)$
- Quantum dynamical map

$$\rho_S(0) \mapsto \rho_S(t) = \Phi(t)\rho_S(0) = \mathrm{Tr}_E(e^{-\frac{i}{\hbar}Ht}(\rho_S(0) \otimes \rho_E)e^{+\frac{i}{\hbar}Ht})$$

Existence of reduced dynamics

Quantum Markov process

- Markov condition
  Separation of time scales $\tau_E \ll \tau_S$
- Semigroup composition law

$$\Phi(s)\Phi(t) = \Phi(t+s) \qquad t, s \geq 0 \quad \Rightarrow \quad \Phi(t) = \exp(\mathcal{L}t)$$

- Quantum dynamical semigroups

[Kossakowski, RMP 1972; Gorini, Kossakowski & Sudarshan, JMP 1976; Lindblad, CMP 1976]

Quantum Markov process fixed by master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t)$$

with $\mathcal{L}$ generator in GKSL form (by now 40 years old!)

Semigroup composition law

Quantum dynamical semigroup

- GKSL generator

$$\mathcal{L}\rho = -\frac{i}{\hbar}[H,\rho] + \sum_k \gamma_k \left[L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right]$$

$$\mathcal{L}\rho = R\rho + \rho R^\dagger + \sum_k \gamma_k L_k \rho L_k^\dagger \quad R = -iH - \frac{1}{2}\sum_k \gamma_k L_k^\dagger L_k$$

- Dyson expansion for exact solution

$$\Phi(t)\rho(0) = \rho(t) = \mathcal{R}(t)\rho(0) + \sum_{n=1}^{\infty} \int_0^t dt_n \ldots \int_0^{t_2} dt_1$$
$$\times \mathcal{R}(t-t_n)\mathcal{J}\mathcal{R}(t_n - t_{n-1})\ldots\mathcal{J}\mathcal{R}(t_1)\rho(0)$$

- Basic building blocks
  CP contraction semigroup $\mathcal{R}(t)\rho = \exp(t\,R)\rho\exp(t\,R^\dagger)$
  CP superoperator $\mathcal{J}\rho = \sum_k \gamma_k L_k \rho L_k^\dagger$
- Both formulations lead to enlarge class of processes amenable to possible quantum counterparts
- Combine viewpoints in order to easily warrant CPT

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Generalized master equation

- Kolmogorov equation for a Markov process

$$\begin{aligned} T_{nm}(t) &= \delta_{nm} e^{-\lambda_m t} + \int_0^t d\tau \sum_k e^{-\lambda_n (t-\tau)} \pi_{nk} \lambda_k T_{km}(\tau) \\ \hat{T}_{nm}(u) &= \delta_{nm} (\lambda_m + u)^{-1} + \sum_k (\lambda_m + u)^{-1} \pi_{nk} \lambda_k \hat{T}_{km}(u) \end{aligned}$$

- Generalize waiting time distribution and survival probability

$$(\lambda_m + u)^{-1} \to \hat{g}_k(u) \quad \pi_{nk}\lambda_k \to \hat{W}_{nk}(u) = \pi_{nk} \hat{f}_k(u)/\hat{g}_k(u)$$

$$f_n(\tau) \geq 0 \qquad \int_0^{+\infty} d\tau\, f_n(\tau) = 1 \qquad g_n(\tau) = 1 - \int_0^{\tau} ds\, f_n(s)$$

- Generalized master equation ($T_{nm} \to P_n$ fixing initial state)

$$\frac{d}{dt} P_n(t) = \int_0^t d\tau \sum_k \left[W_{nk}(\tau) P_k(t-\tau) - W_{kn}(\tau) P_n(t-\tau) \right]$$

[Feller, 1964; Gillespie, 1977; Esposito & Lindenberg, 2008; Breuer & B.V., PRL 2008; Breuer & B.V., PRA 2009]

Generalized master equation

- Basic structure in Laplace transform

$$u\hat{P}_n(u) - P_n(0) = \sum_m \left[\pi_{nm} \frac{\hat{f}_m(u)}{\hat{g}_m(u)} - \delta_{nm} \left(\frac{1}{\hat{g}_m(u)} - u \right) \right] \hat{P}_m(u)$$

- Starting point for possible quantum generalizations

$$\begin{aligned} u\hat{\rho}(u) - \rho(0) &= \left\{ \mathcal{O}\left[\pi \frac{\hat{f}(u)}{\hat{g}(u)} \right] - \mathcal{O}\left[\frac{1}{\hat{g}(u)} - u \right] \right\} \hat{\rho}(u) \\ &= \hat{\mathcal{K}}(u)\hat{\rho}(u) \end{aligned}$$

where $\mathcal{O}[\cdot]$ stands for operator replacement

- Jump and evolution map

π	$\rightarrow$	$\mathcal{E}$	CPT map
$f(t)$	$\rightarrow$	$f(t)\mathcal{F}(t)$	$\mathcal{F}(t)$ CPT maps
$g(t)$	$\rightarrow$	$g(t)\mathcal{G}(t)$	$\mathcal{G}(t)$ CPT maps, s.t. $\mathcal{G}(0) = \mathbb{1}$

[Breuer & B.V., PRL 2008; Chruscinski & Kossakowski, PRA 2016; B.V., PRL 2016]

Operator correspondence

Expressions of memory kernel I: collisional models

- Scheme I of operator ordering

$$\mathcal{O}\left[\pi\frac{\hat{f}(u)}{\hat{g}(u)}\right] \quad \to \quad \frac{1}{\widehat{g\mathcal{G}}(u)}\widehat{f\mathcal{F}}(u)\mathcal{E}$$

- Evolution map

$$\widehat{\Phi}(u)\rho(0) = \hat{\rho}(u) = (u-\widehat{\mathcal{K}}(u))^{-1}\rho(0) = [\mathbb{1}-\widehat{f\mathcal{F}}(u)\mathcal{E}]^{-1}\widehat{g\mathcal{G}}(u)\rho(0)$$

- Time domain

$$\rho(t) = \sum_{n=0}^{\infty}\Big(*^{n}(f\mathcal{F}\mathcal{E}) * (g\mathcal{G})\Big)(t)\rho(0)$$

composition of CP maps immediately warranting CP

Expressions of memory kernel I: collisional models

- General request for trace preservation read from kernel

$$\frac{\mathrm{d}}{\mathrm{dt}} \mathrm{Tr}\,\{g(t)\mathcal{G}(t)\rho\} = -\,\mathrm{Tr}\{M(t)\rho\}$$

$$\mathrm{Laplace}(M(t);u) = \widehat{g\mathcal{G}}(u)^{-1}\widehat{f\mathcal{F}}(u)\mathcal{E}\widehat{g\mathcal{G}}(u)$$

- Trace preservation for $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ trace preserving leads to

$$\frac{1}{u}\mathbb{1} = \frac{\widehat{g}(u)}{1-\widehat{f}(u)}\mathbb{1} \longrightarrow \frac{d}{dt}g(t) = -f(t)$$

[Ciccarello & al., PRA(R) 2013; B.V., PRA(R) 2013; Lorenzo & al., PRA 2016; B.V., PRL 2016]

Warranting of CPT

Expressions of memory kernel II: collisional models

- Scheme II of operator ordering

$$\mathcal{O}\left[\pi\frac{\hat{f}(u)}{\hat{g}(u)}\right] \to \mathcal{E}\widehat{f\mathcal{F}}(u)\frac{1}{\widehat{g\mathcal{G}}(u)}$$

- Time domain

$$\rho(t) = \sum_{n=0}^{\infty}\Big((g\mathcal{G}) *^{n} (\mathcal{E}f\mathcal{F})\Big)(t)\rho(0)$$

- General request for trace preservation read from kernel

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{Tr}\left\{g(t)\mathcal{G}(t)\rho\right\} = -\mathrm{Tr}\{\mathcal{E}f(t)\mathcal{F}(t)\rho\}$$

previous simple requirement for $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ trace preserving

Alternative warranting of CPT

Expressions of kernel III: CTQRW and micromaser

- Special choice of evolution maps

$$\begin{array}{ccclr} f(t) & \rightarrow & f(t)\mathrm{e}^{\mathcal{L}t} & \mathcal{L}\ \text{GKSL generator} & [\hat{f}(u) \rightarrow \hat{f}(u-\mathcal{L})] \\ g(t) & \rightarrow & g(t)\mathrm{e}^{\mathcal{L}t} & \mathcal{L}\ \text{GKSL generator} & [\hat{g}(u) \rightarrow \hat{g}(u-\mathcal{L})] \\ \pi & \rightarrow & \mathcal{E} & \text{arbitrary CPT map} & \end{array}$$

- Scheme of operator ordering

$$\mathcal{O}\left[\pi\frac{\hat{f}(u)}{\hat{g}(u)}\right] = \frac{\hat{f}(u-\mathcal{L})}{\hat{g}(u-\mathcal{L})}\mathcal{E}$$

setting $\hat{k}(u) = \hat{f}(u)/\hat{g}(u)$ leads to

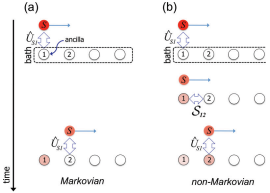
$$\frac{\mathrm{d}}{\mathrm{dt}}\rho(t) = \mathcal{L}\rho(t) + \int_0^t \mathrm{d}\,\tau \mathrm{e}^{\mathcal{L}(t-\tau)} k(t-\tau)(\mathcal{E}-\mathbb{1})\rho(\tau)$$

- QDS for $\mathcal{E} \rightarrow \mathbb{1}$ independently of $f(t)$: $\Phi(t) = e^{\mathcal{L}t}$
- CTQRW for $\mathcal{L} \rightarrow 0$ and general $f(t)$: $\Phi(t) = \sum p_k(t)\mathcal{E}^k$

[Cresser, PRA 1992; Herzog, PRA 1995; Budini, PRA 2004 & 2013; B.V., PRA(R) 2013 & IJQI 2014]

Collisional models

- Dynamics modelled through repeated collisions described as instantaneous transformations
- Memory effects embodied in correlations between parts of the environment involved in single collisions



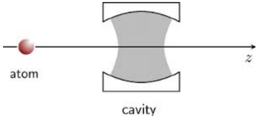
[Rybar & al., JPB 2012; Giovannetti & al., JPB 2012; Ciccarello & al., PRA(R) 2013; Lorenzo & al., PRA 2016]

- Collisional model with memory:
  $f(t) \to$ exponential $\quad f(t) = \Gamma e^{-\Gamma t}, g(t) = e^{-\Gamma t} \quad \mathcal{E} \to \mathbb{1}$
- Microscopic derivation from continuous limit of collision with ancillas which interact through swap operation

Piecewise description of two-level system dynamics

Micromaser model

- Dynamics modelled through repeated passages of excited atoms described as fixed transformations $\mathcal{E}$ together with Markovian dissipation in between $\mathcal{F}(t) = \mathcal{G}(t) = \mathrm{e}^{\mathcal{L}t}$
- Memory effects embodied in distribution in time of atom arrivals



[Cresser, PRA 1992; Herzog, PRA 1995; Cresser, QS Optics 1996]

- Ordinary renewal process for atom arrivals

$$p_n(t_n, \ldots, t_1) = g(t - t_n) \ldots f(t_2 - t_1) f(t_1)$$

Piecewise description of field dynamics

Summary & Acknowledgments

Summary & Outlook

- Completely positive non semigroup evolutions
- Classes of possibly non-Markovian evolutions
- Uinfied viewpoint referring to classical processes
- Different structures arising from quantum correspondence due to operator ordering

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