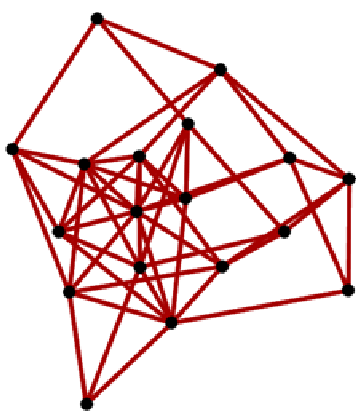
Estimating connectivity of complex networks

Johannes Nokkala,
University of Turku
With Sabrina Maniscalco and Jyrki Piilo

Networks are modeled by graphs

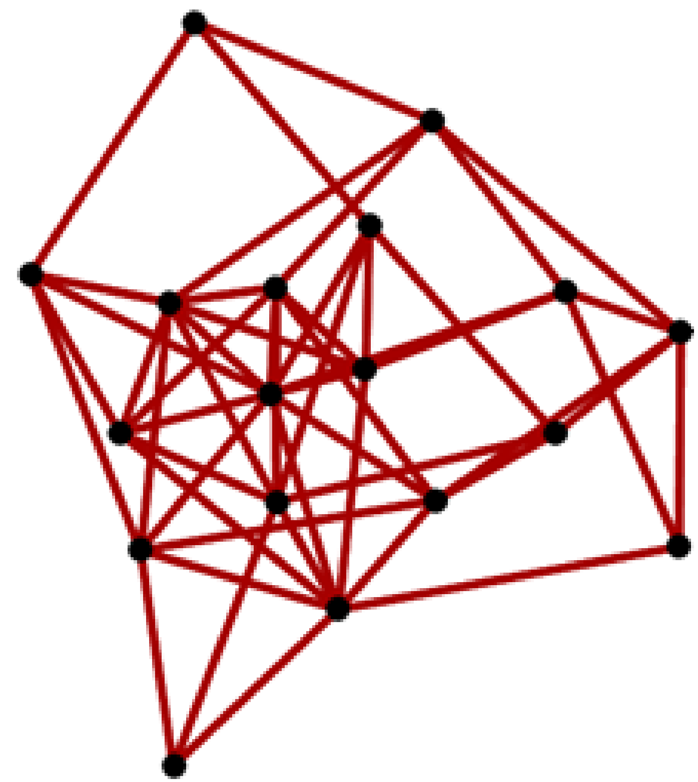
- **Network**: composed of interacting or related units, e.g. social network of scientists
- **Graph**: skeleton of a network, abstract mathematical object, e.g. collaboration graph



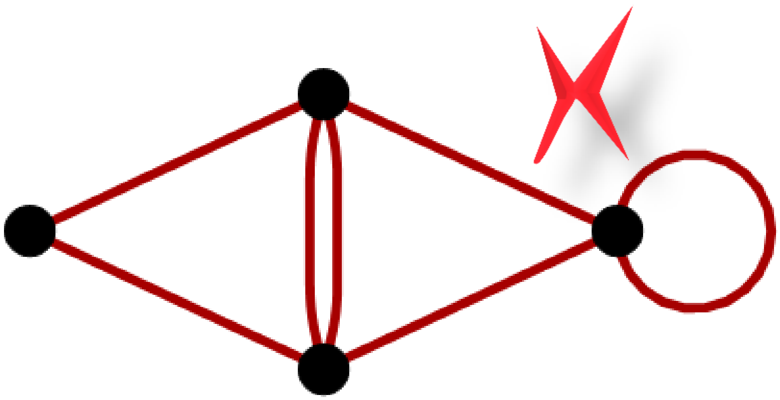
$$\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A},$$
$$L_{ij} = \delta_{ij} d_i - (1 - \delta_{ij}) a_{ij}$$

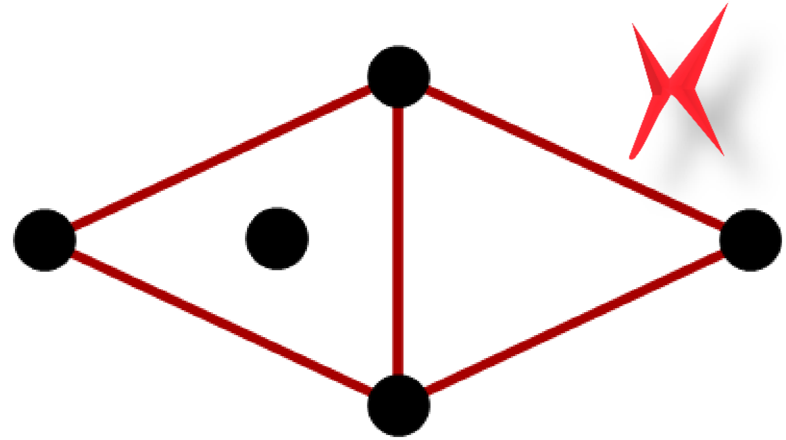
Research question

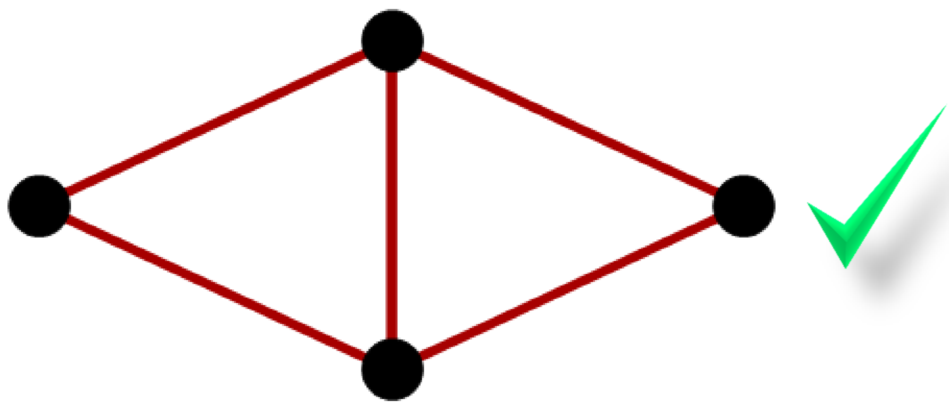
Given the eigenvalues of $\boldsymbol{L}$, what can be said about d_i?



Simple connected graphs!







Connectivity estimation

- Let λ_i be the eigenvalues ordered from largest to smallest

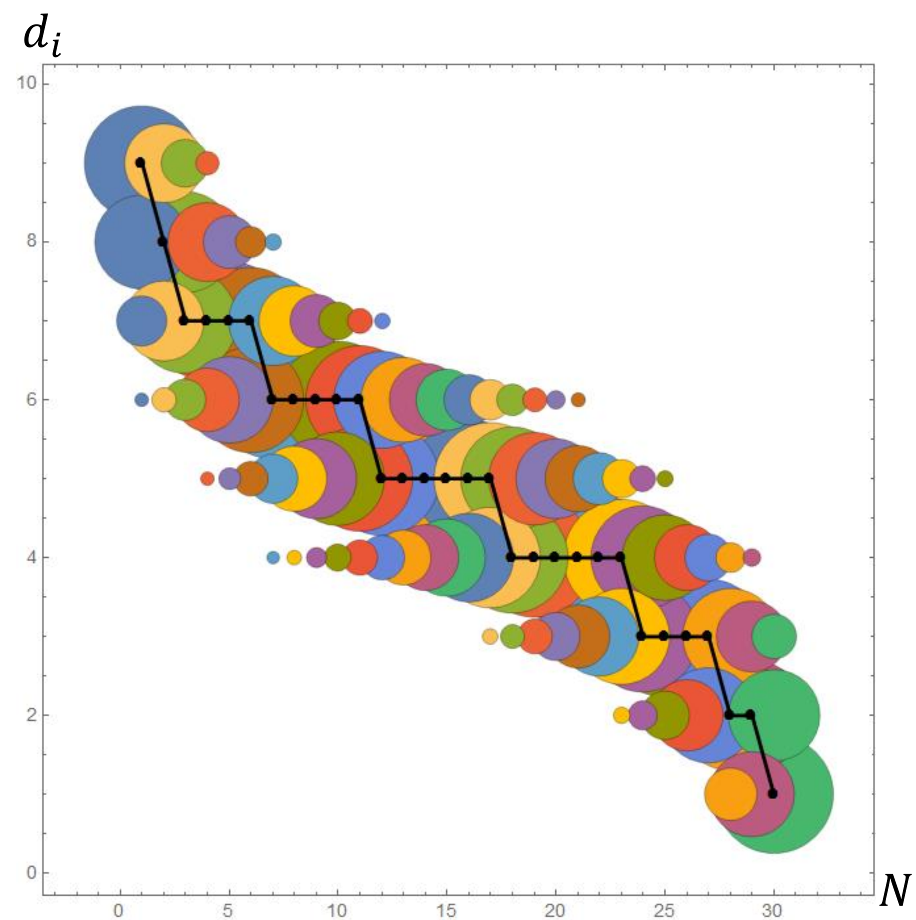
$$\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} d_i \,, \qquad \sum_{i=1}^{N} ({\lambda_i}^2 - \lambda_i) = \sum_{i=1}^{N} {d_i}^2 \,, \qquad \sum_{i=1}^{n<N} \lambda_i \geq 1 + \sum_{i=1}^{n<N} d_i \,.$$

- Also bounds on d_{min}, d_{max}
- Mean and variance of bounded sequence $\boldsymbol{d} = \{d_1, d_2, \dots, d_N\}$ determined

Spectra of graphs
(Springer, New York, 2011)

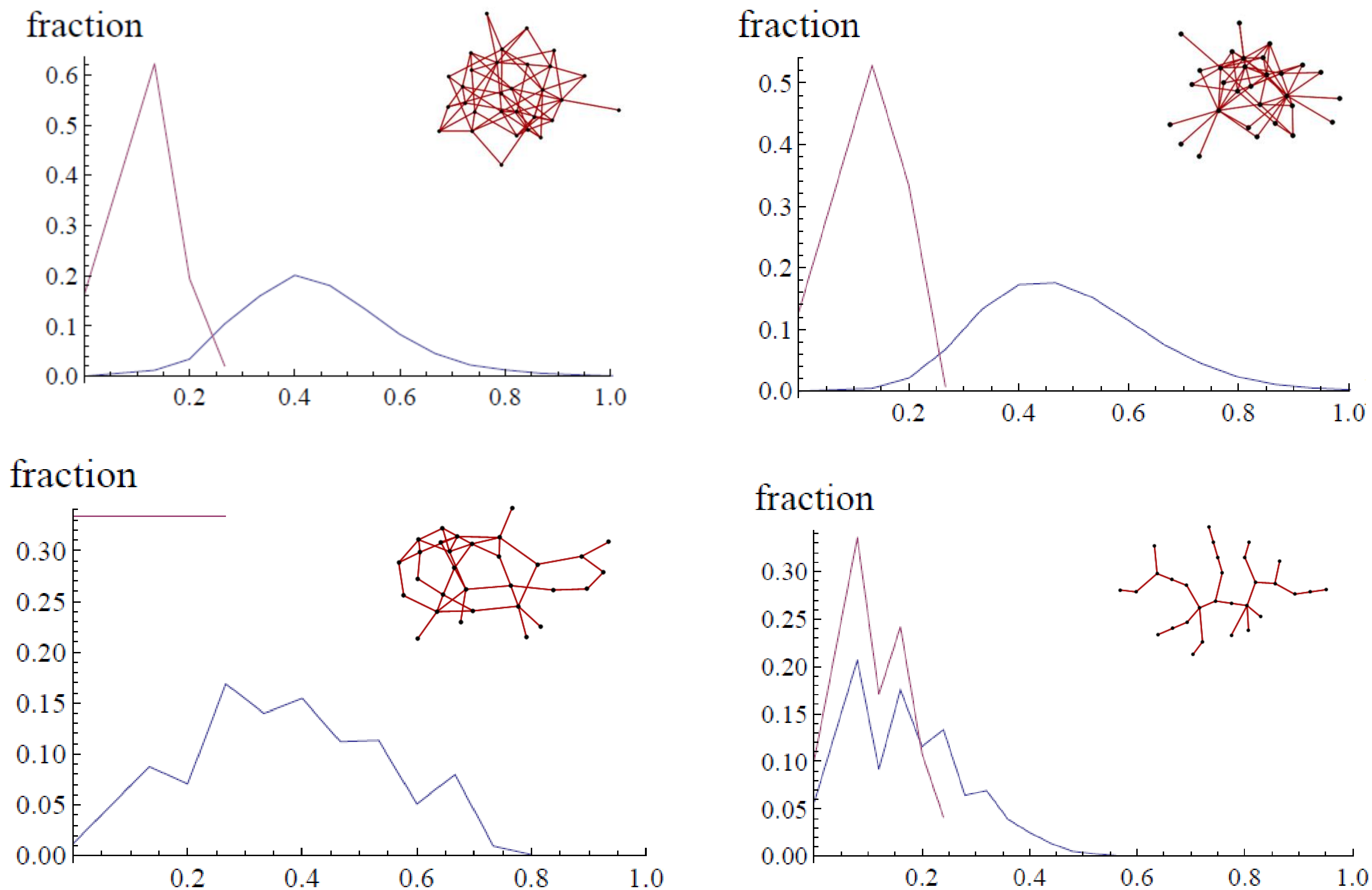
Connectivity estimation

- Finite number of possible sequences
- The one closest to their mean is the estimate $\boldsymbol{d}'$
- Unique solution: chain, regular, completely connected



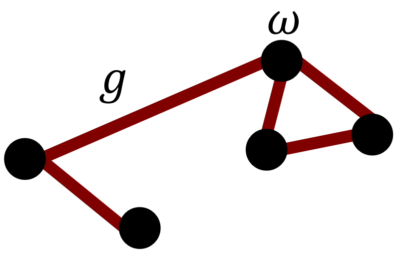
Distance between d and d'

- Figure of merit: $\sum_i^N |d_i - d'_i|/N = \|\boldsymbol{d} - \boldsymbol{d}'\|_1/N$



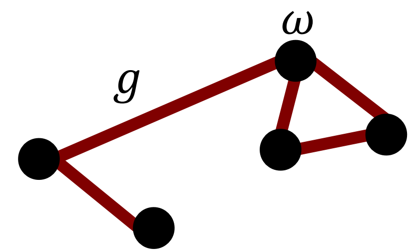
Identical oscillators, uniform harmonic couplings

$$H_E = \frac{\boldsymbol{p}^T \boldsymbol{p}}{2} + \boldsymbol{q}^T (\omega^{\mathbf{2}} \boldsymbol{I} + g \boldsymbol{L}) \boldsymbol{q}$$

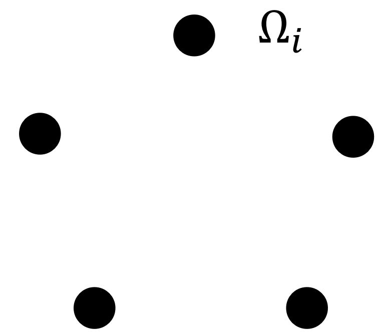


Identical oscillators, uniform harmonic couplings

$$H_E = \frac{\boldsymbol{p}^T\boldsymbol{p}}{2} + \boldsymbol{q}^T(\omega^{\mathbf{2}}\boldsymbol{I} + g\boldsymbol{L})\boldsymbol{q}$$

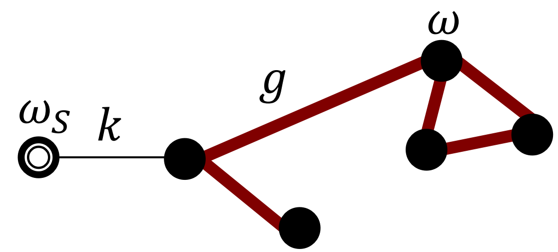


$$H_E = \sum_{i=1}^{N}({P_i}^2 + {\Omega_i}^2{Q_i}^2)/2$$

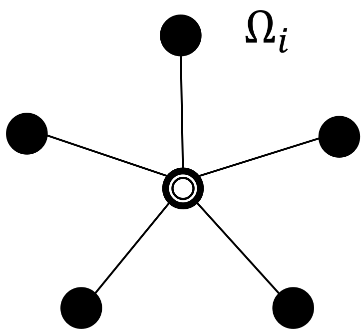


Identical oscillators, uniform harmonic couplings

$$H_E = \frac{\boldsymbol{p}^T\boldsymbol{p}}{2} + \boldsymbol{q}^T(\omega^{\mathbf{2}}\boldsymbol{I} + g\boldsymbol{L})\boldsymbol{q}$$



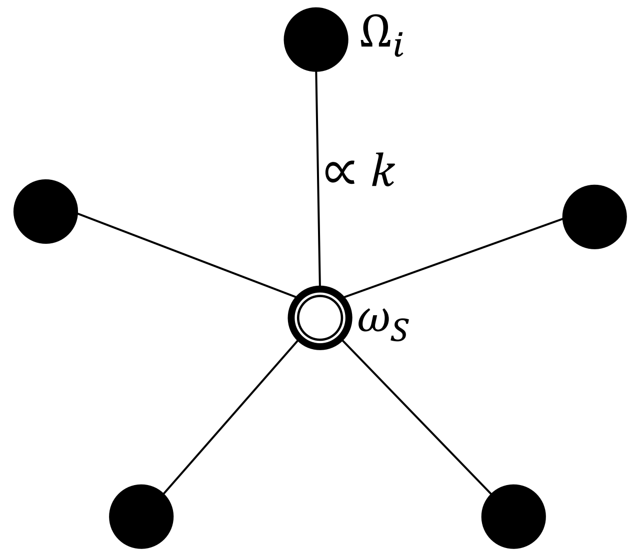
$$H_E = \sum_{i=1}^{N}(P_i{}^2 + \Omega_i{}^2 Q_i{}^2)/2$$



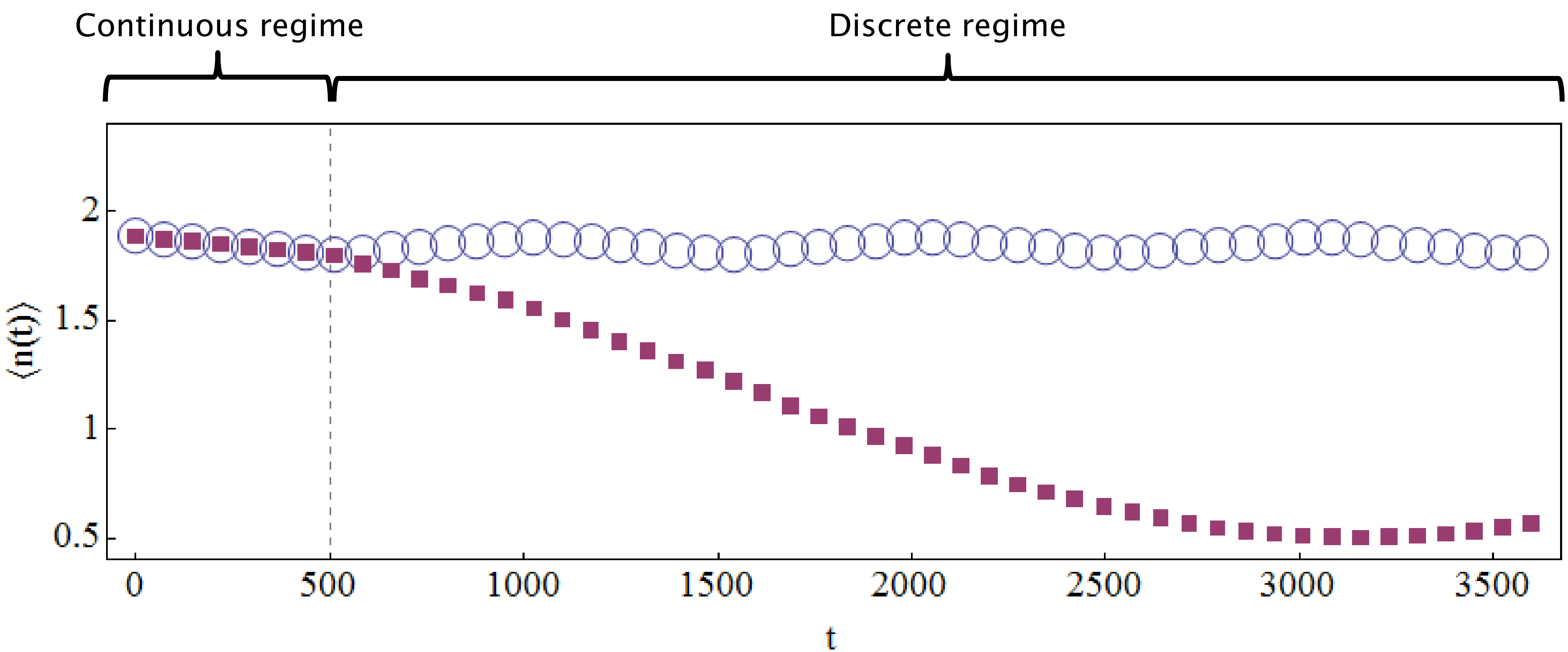
$$H_S = (p_S^2 + \omega_S^2 q_S^2)/2, \quad H_I = -kq_S q_j$$

Recovering the eigenvalues

- $\lambda_i = \left(\Omega_i{}^2 - \omega^2\right)/g$
- Generic network → probe interacts with and resolves all modes
- Weak k → sensitivity to Ω_i

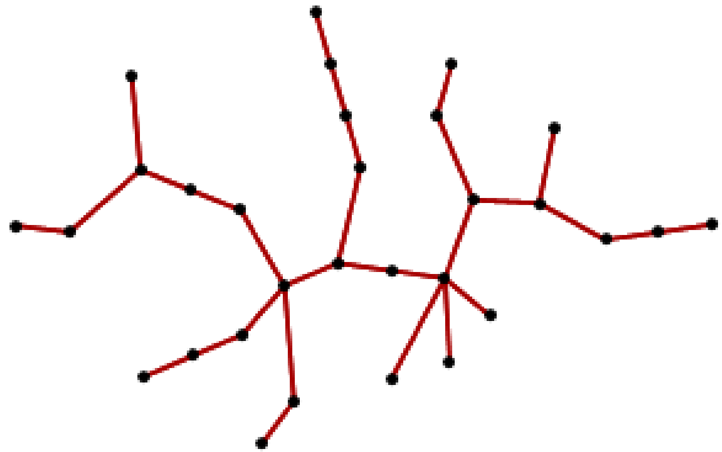


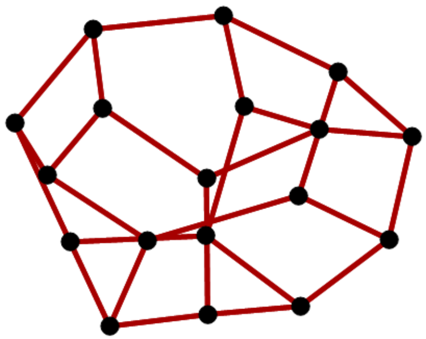
Probing of Ω_i



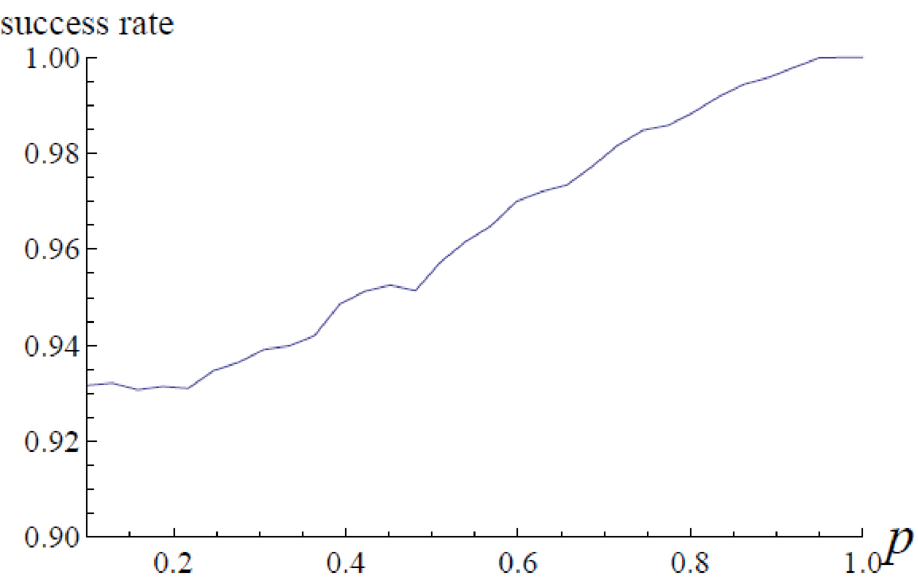
Coupling strength estimation

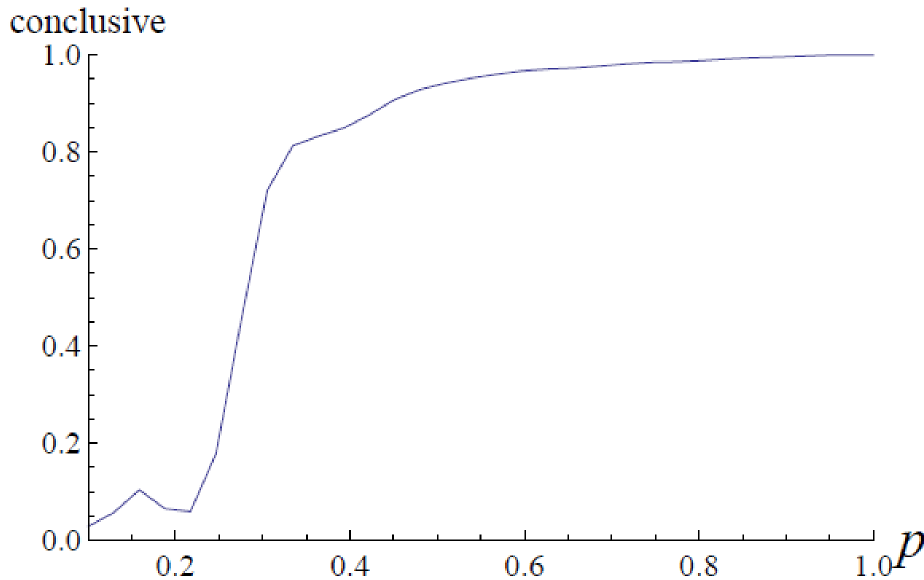
- Suppose g unknown
- $g\lambda_i = {\Omega_i}^2 - \omega^2$
- $2(N-1) \leq \sum d_i \leq N(N-1)$ $\rightarrow$ bounds on g
- $\sum d_i, \sum {d_i}^2$ must be even $\rightarrow$ finite set of possible values for g, integer fractions will work too $\rightarrow$ choose largest possible value as estimate
- Guaranteed to work for trees, regular networks





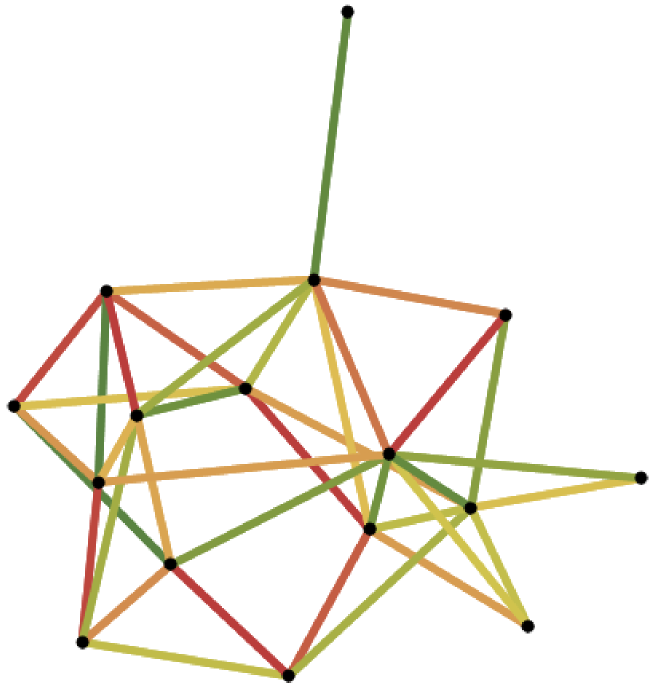
Coupling strength estimation





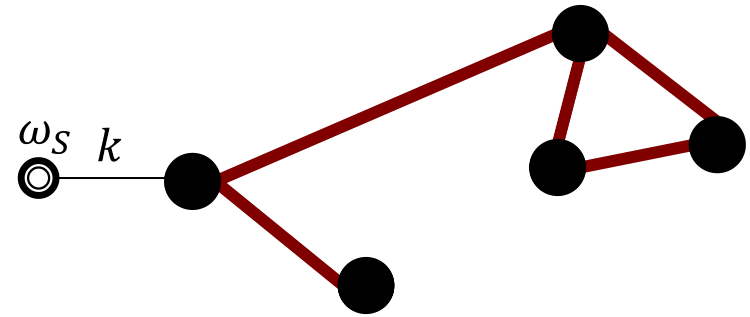
Nonuniform coupling strength

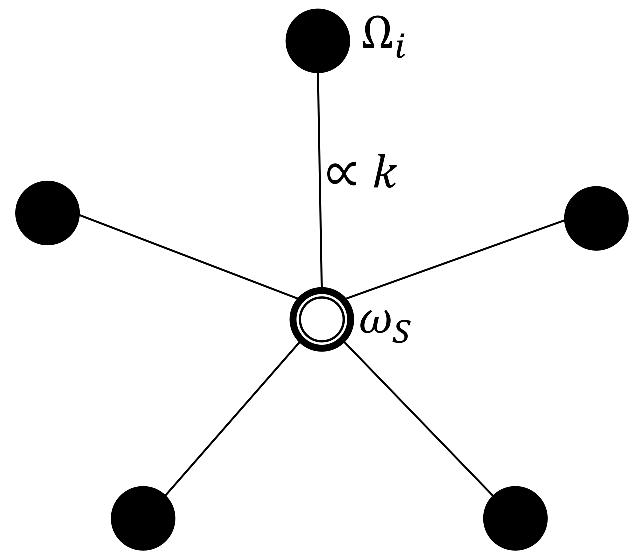
- Now $H_E = \frac{\boldsymbol{p}^T\boldsymbol{p}}{2} + \boldsymbol{q}^T(\omega^{\mathbf{2}}\boldsymbol{I} + \boldsymbol{L}_w)\boldsymbol{q}$, where $\boldsymbol{L}_w$ the weighted $\boldsymbol{L}$
- In general, scheme not applicable



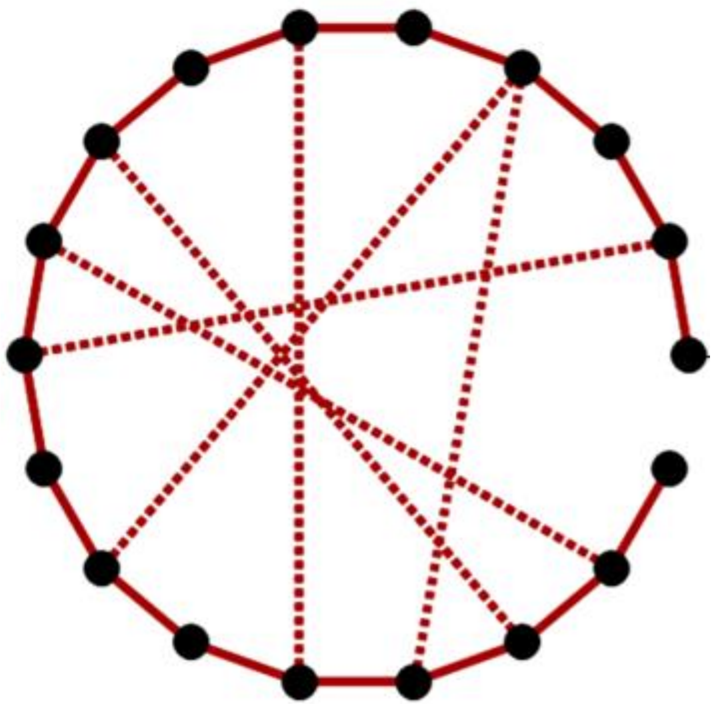
Full access

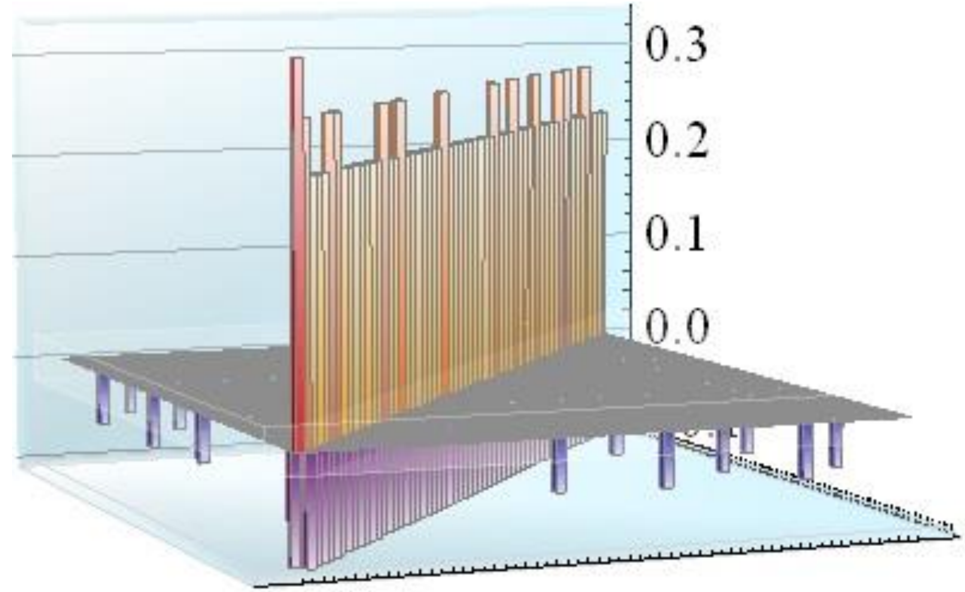
- Coupling strengths to eigenmodes give the mapping back to network picture
- Can be extracted from, e.g., rate of energy exchange
- Single and pairwise couplings needed

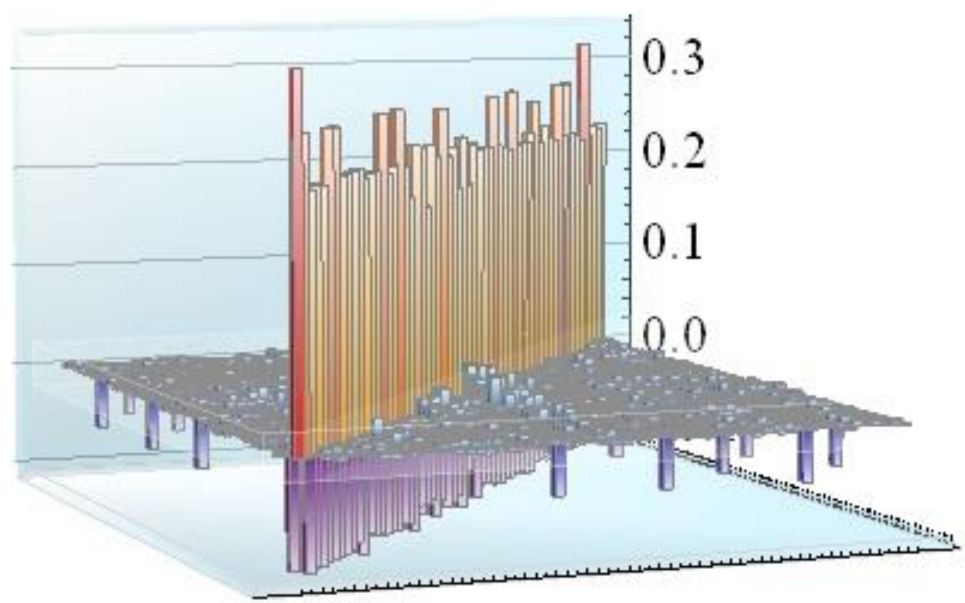




Scientific Reports **6**, 26861 (2016)

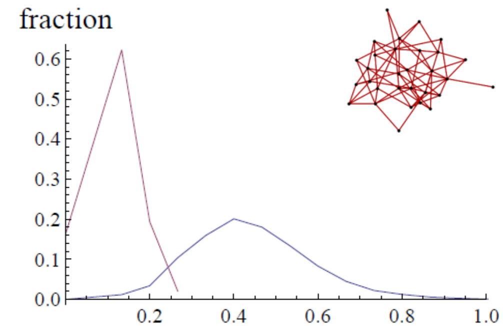


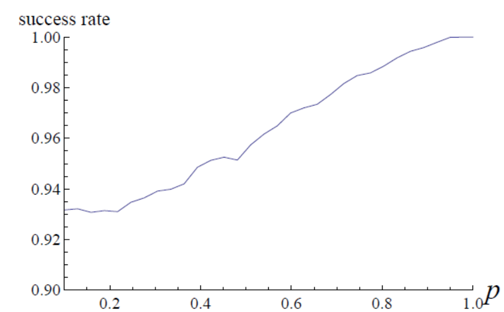

0.3
0.2
0.1
0.0

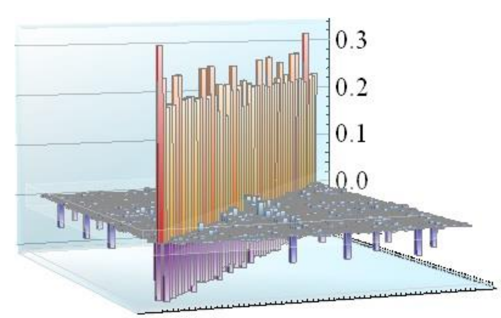

0.3
0.2
0.1
0.0

Summary and outlook

- d and a uniform g can be estimated from any single node
- Full access, full reconstruction
- Proposal for an experimental implementation of quantum networks in preparation







Thank you!