Non-Markovian Quantum Dynamics of Open Systems

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Non-Markovian Quantum Dynamics

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Contents

- Open systems and quantum dynamical maps
- Definition of quantum non-Markovianity: Trace distance, distinguishability of quantum states and information flow
- Generalized definition: Helstrom matrix, divisibility and classification of quantum processes
- Summary

References:

HPB, E.-M. Laine, J. Piilo, B. Vacchini, arXiv:1505.01385 [quant-ph]

S. Wißmann, B. Vacchini, HPB, arXiv:1507.08867 [quant-ph]

Open systems and quantum dynamical maps

$$
\begin{array}{ccc}
\rho(0) = \rho_S(0) \otimes \rho_E & \xrightarrow{\text{unitary evolution}} & \rho(t) = U_t \big[\rho_S(0) \otimes \rho_E \big] U_t^\dagger \\
\mathrm{tr}_E \big\downarrow & & \big\downarrow \mathrm{tr}_E \\
\rho_S(0) & \xrightarrow{\text{dynamical map}} & \rho_S(t) = \Phi_t \, \rho_S(0)
\end{array}
$$

Quantum dynamical (CPT) map:

$$\rho_S(0) \longrightarrow \rho_S(t) = \Phi_t \, \rho_S(0) = \mathrm{tr}_E \left\{ U_t \big[\rho_S(0) \otimes \rho_E \big] U_t^\dagger \right\}$$

Quantum process: One-parameter family of dynamical maps:

$$\{ \Phi_t \,|\, \Phi_0 = I, \; 0 \leq t \leq T \}$$

Example: Quantum dynamical semigroup

Dynamical semigroup:

$$\Phi_{t_1+t_2} = \Phi_{t_2}\Phi_{t_1} \Longrightarrow \Phi_t = e^{\mathcal{L}t}$$

with Lindblad generator:

$$\mathcal{L}\rho_S = -i\left[H_S, \rho_S\right] + \sum_i \gamma_i \left[A_i \rho_S A_i^\dagger - \frac{1}{2}\left\{A_i^\dagger A_i, \rho_S\right\}\right] \qquad \gamma_i \geq 0$$

$\Longrightarrow$ **Quantum master equation:**

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}\rho_S(t)$$

Microscopic derivation: Separation of time scales:

$$\tau_E \ll \tau_R$$

Non-Markovian quantum dynamics

- Standard Markov condition $\tau_E \ll \tau_R$ violated
- Correlation functions of higher order play important role
- Finite revival times (finite reservoir)
- Correlations and entanglement in the initial state

Goals:

- What is the **key feature** of non-Markovian dynamics?
- How can one **define** non-Markovianity without reference to specific representation, master equation, approximation etc.?
- **Memory and information flow?**
- Construction of a **measure** for non-Markovianity?
- Experimentally **measurable** quantity?

Distance between quantum states

Trace norm of an operator A:

$$||A|| = \mathrm{tr}|A| \qquad |A| = \sqrt{A^\dagger A}$$

For selfadjoint operators:

$$A = \sum_i a_i \, |i\rangle \langle i|$$

$$||A|| = \sum_i |a_i|$$

Trace distance between quantum states ρ_1 and ρ_2:

$$D(\rho_1, \rho_2) = \frac{1}{2}||\rho_1 - \rho_2|| = \frac{1}{2}\mathrm{tr}|\rho_1 - \rho_2|$$

Properties of the trace distance

- **Metric on state space with $0 \le D(\rho_1, \rho_2) \le 1$**

$$\rho_1 = \rho_2 \iff D = 0$$
$$\rho_1 \perp \rho_2 \iff D = 1$$

- **CPT maps Φ are contractions for the trace distance:**

$$D(\Phi\rho_1, \Phi\rho_2) \le D(\rho_1, \rho_2)$$

- **Representation through maximum over all projections or positive operators $\Pi \le I$:**

$$D(\rho_1, \rho_2) = \max_{\Pi} \mathrm{tr}\Big\{\Pi(\rho_1 - \rho_2)\Big\}$$

Physical interpretation

Alice

Preparation:

ρ_1 or ρ_2

with $p_1 = p_2 = 1/2$

Bob

Measurement:

ρ_1 or ρ_2 ?

success with

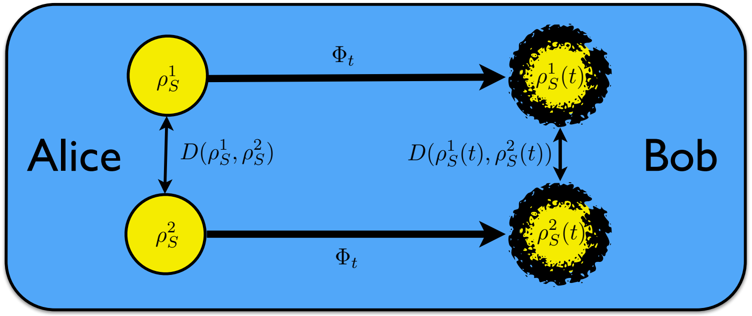
prob = $(1+D(\rho_1,\rho_2))/2$

$\Rightarrow D(\rho_1, \rho_2) =$ **Measure for distinguishability of** ρ_1 **and** ρ_2

Example: $\rho_1 \perp \rho_2 \Rightarrow$ **prob** $= 1$

Dynamics of initial state pairs

$$\rho_S^{1,2}(0) \longrightarrow \rho_S^{1,2}(t) = \Phi_t \, \rho_S^{1,2}(0)$$



Trace distance evolution:

$$D(t) = D(\rho_S^1(t), \rho_S^2(t)) \leq D(\rho_S^1(0), \rho_S^2(0))$$

Monotonic decrease of $D(t)$: Decrease of distinguishability:

Flow of information from system to environment

$D(t)$ temporarily increases: Increase of distinguishability:

Flow of information from environment back to system

Definition of quantum non-Markovianity

Definition: A quantum process Φ_t is **non-Markovian** iff

$$\sigma(t) = \frac{d}{dt} D(\rho_S^1(t), \rho_S^2(t)) > 0$$

for some initial state pair $\rho_S^{1,2}(0)$ and some time $t > 0$

- Increase of the distinguishability of the states $\rho_S^{1,2}$
- Flow of information from environment back to system
- Environment acts as memory

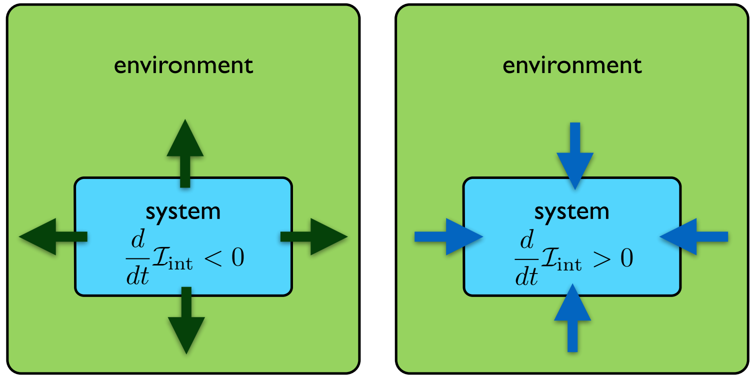
Measure for non-Markovianity

$$\mathcal{N}(\Phi) = \max_{\rho_S^{1,2}(0)} \int_{\sigma>0} dt\ \sigma(t) \qquad \sigma(t) = \frac{d}{dt} D(\rho_S^1(t), \rho_S^2(t))$$

- Measures total backflow of information
- Summation over all time intervals in which $\sigma > 0$
- Maximum over all pairs of initial states $\rho_S^{1,2}(0)$
- $\mathcal{N}(\Phi) > 0 \Longleftrightarrow$ process non-Markovian
- Possible values: $0 \leq \mathcal{N}(\Phi) \leq +\infty$

HPB, E.-M. Laine, J. Piilo, Phys. Rev. Lett. 103, 210401 (2009)

Information flow



Information inside open system:

$$\mathcal{I}_{\rm int}(t) = D(\rho_S^1(t), \rho_S^2(t))$$

Information outside open system:

$$\mathcal{I}_{\rm ext}(t) = D(\rho^1(t), \rho^2(t)) - D(\rho_S^1(t), \rho_S^2(t)) \geq 0$$

Information flow:

$\frac{d}{dt}\mathcal{I}_{\rm int}(t) < 0$ **open system looses information (Markovian)**

$\frac{d}{dt}\mathcal{I}_{\rm int}(t) > 0$ **open system gains information (non-Markovian)**

Information flow

Conservation of information:

$$\mathcal{I}_{\rm int}(t) + \mathcal{I}_{\rm ext}(t) = \mathcal{I}_{\rm int}(0) = \textbf{const}$$

General inequality based on properties of the trace distance:

$$\mathcal{I}_{\rm ext}(t) \leq D(\rho^1(0), \rho_S^1(0) \otimes \rho_E^1(0)) + D(\rho^2(0), \rho_S^2(0) \otimes \rho_E^2(0)) + D(\rho_E^1(0), \rho_E^2(0))$$

Interpretation: Information outside the open system implies
system-environment correlations or different environmental states

Generalizing the trace distance measure

- **Alice prepares two quantum states $\rho_S^{1,2}$ with probabilities $p_{1,2}$ which need not be equal (biased preparation of states)**

- **The maximal probability for a successful state discrimination Bob can achieve by an optimal strategy is given by**

$$P_{\max} = \frac{1}{2}\{1 + ||\Delta||\}$$

where Δ is the Helstrom matrix:

$$\Delta = p_1\rho_S^1 - p_2\rho_S^2$$

D. Chruscinski, A. Kossakowski, A. Rivas, Phys. Rev. A 83, 052128 (2011)

Generalized definition of non-Markovianity

Replacing the trace distance by the norm of the Helstrom matrix leads to generalized non-Markovianity measure:

$$\mathcal{N}(\Phi) = \max_{||\Delta||=1} \int_{\sigma>0} dt\, \sigma(t) \qquad \sigma(t) = \frac{d}{dt}||\Phi_t \Delta||$$

where the maximum is taken over all Helstrom matrices with unit trace norm:

$$\Delta = p_1 \rho_S^1 - p_2 \rho_S^2 \qquad ||\Delta|| = 1$$

S. Wißmann, B. Vacchini, HPB, arXiv:1507.08867 [quant-ph]

Generalized definition of non-Markovianity

Advantages of this approach:

- **Markovianity is equivalent to P-divisibility of dynamical maps**
- **Orthogonality of optimal state pairs and local representation**
- **It leads to a general classification of quantum processes in open systems**
- **It yields direct connection to notion of a classical Markov process**

Divisibility of quantum processes

If inverse of dynamical maps exists we define for all $t \geq s \geq 0$:

$$\Phi_{t,s} = \Phi_t \, \Phi_s^{-1} \implies \Phi_t = \Phi_{t,s} \, \Phi_s$$

Definition:

$\Phi_{t,s}$ **completely positive** $\iff$ **process is CP-divisible**

$\Phi_{t,s}$ **positive** $\iff$ **process is P-divisible**

For generalized definition based on Helstrom matrix:

Markovianity is equivalent to P-divisibility

Proof: By use of Kossakowski theorem (Rep. Math. Phys. 3, 247 (1972))

Relation to time-local master equation

Most general structure of time-local master equation:

$$\frac{d}{dt}\rho_S = \mathcal{K}_t\,\rho_S$$

$$= -i\,[H_S(t),\rho_S] + \sum_i \gamma_i(t)\left[A_i(t)\rho_S A_i^\dagger(t) - \frac{1}{2}\left\{A_i^\dagger(t)A_i(t),\rho_S\right\}\right]$$

- **Dynamics is CP-divisible if and only if**

$$\gamma_i(t) \geq 0$$

- **Dynamics is P-divisible if and only if**

$$W_{nm}(t) \equiv \sum_i \gamma_i(t)\,|\langle n|A_i(t)|m\rangle|^2 \geq 0$$

Relation to time-local master equation

Pauli master equation in instantaneous eigenbasis of $\rho_S(t)$:

$$\frac{d}{dt}P_n(t) = \sum_m \left[W_{nm}(t)P_m(t) - W_{mn}(t)P_n(t) \right]$$

where

$$W_{nm}(t) = \sum_i \gamma_i(t) |\langle n(t)|A_i(t)|m(t)\rangle|^2$$

$\Longrightarrow$ can be interpreted as **Chapman-Kolmogorov equation of a classical Markov process if and only if**

$$W_{nm}(t) \geq 0$$

Conclusion:

quantum Markovian $\Longleftrightarrow$ P-divisible $\Longrightarrow$ classical Markovian

Local representation and classification of dynamics

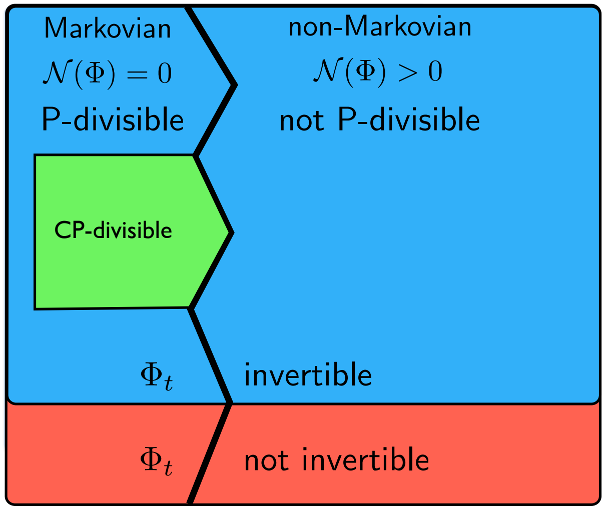
- **Orthogonality of optimal state pairs:**

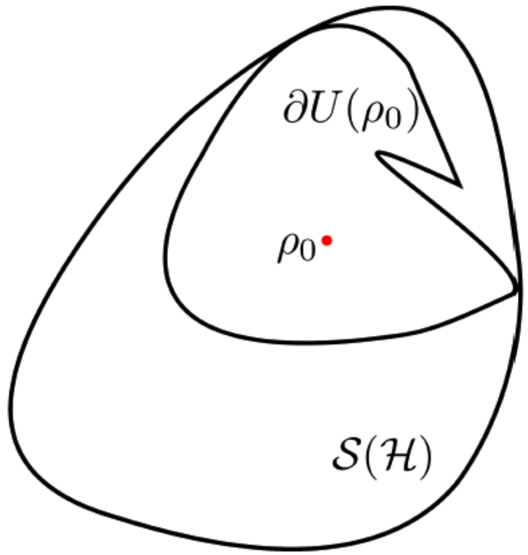
$$\mathcal{N}(\Phi) = \max_{p_{1,2},\, \rho_S^1 \perp \rho_S^2} \int_{\sigma>0} dt\, \sigma(t) \qquad \sigma(t) = \frac{d}{dt}||\Phi_t \Delta||$$

- **Local representation:**

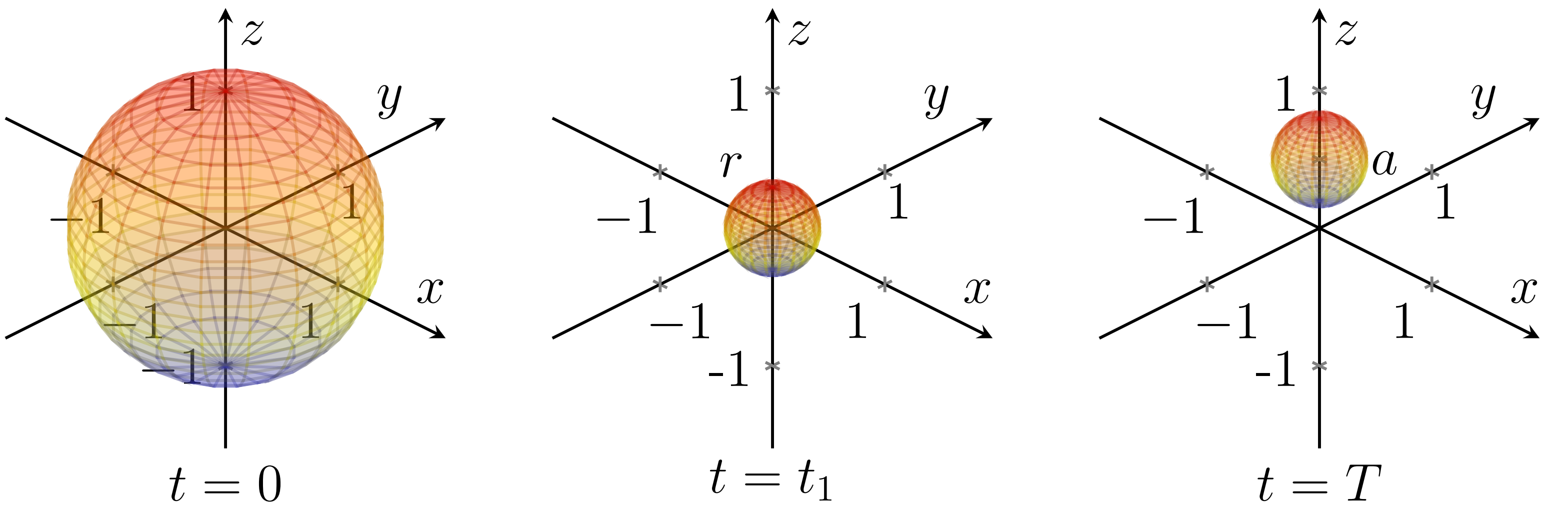
$$\mathcal{N}(\Phi) = \max_{p_{1,2},\, \rho \in \partial U(\rho_0)} \int_{\sigma>0} dt\, \sigma(t) \qquad \sigma(t) = \frac{d}{dt}||\Phi_t \Delta||/||\Delta||$$

- **Classification of quantum processes:**





Example



First phase ($0 \leq t \leq t_1$): **Isotropic contraction** **of Bloch sphere:**

$$\vec{v} \mapsto r\,\vec{v} \qquad 0 < r < 1 \qquad \textbf{CP-divisible}$$

Second phase ($t_1 \leq t \leq T$): **Uniform translation:**

$$\vec{v} \mapsto \vec{v} + \vec{a} \qquad 0 < |\vec{a}| \leq 1 - r \qquad \textbf{not positive}$$

The combined process is CP but not P-divisible

Example

Norm of Helstrom matrix ($p_1 \geq p_2$):

$$\|\Delta\| = \begin{cases} p_1 - p_2 & \text{for } p_1 - p_2 > |\vec{w}| \\ |\vec{w}| & \text{for } p_1 - p_2 \leq |\vec{w}| \end{cases} \qquad \vec{w} = p_1\vec{v}_1 - p_2\vec{v}_2$$

For $p_1 = p_2 = \frac{1}{2}$ (unbiased case):

$$\|\Delta\| = |\vec{w}| = \frac{1}{2}|\vec{v}_1 - \vec{v}_2| = \text{trace distance}$$

$\Longrightarrow$ impossibility to detect non-Markovianity due to translational invariance of trace distance

Non-Markovianity measure based on Helstrom matrix:

$$\mathcal{N}(\Phi) = r\,|\vec{a}|$$

$\Longrightarrow$ directly proportional to length of translation vector $\vec{a}$

Summary

- **Definition of quantum non-Markovianity:**

$$\sigma(t) = \frac{d}{dt} D(\rho_S^1(t), \rho_S^2(t)) > 0$$

$\Longrightarrow$ **Reversed flow of information from environment to system**

- **Measure for non-Markovianity:**

$$\mathcal{N}(\Phi) = \max_{\rho_S^{1,2}(0)} \int_{\sigma > 0} dt \; \sigma(t)$$

$\Longrightarrow$ **Measure for total backflow of information**

Summary

- **Generalized defintion based on Helstrom matrix Δ:**

$$\mathcal{N}(\Phi) = \max_{||\Delta||=1} \int_{\sigma>0} dt\, \sigma(t) \qquad \sigma(t) = \frac{d}{dt}||\Phi_t \Delta||$$

$\Longrightarrow$ **Markovianity equivalent to P-divisibility, local representation and general classification**

Review article:

HPB, E.-M. Laine, J. Piilo, B. Vacchini, arXiv:1505.01385 [quant-ph]

Further details and proofs:

S. Wißmann, B. Vacchini, HPB, arXiv:1507.08867 [quant-ph]